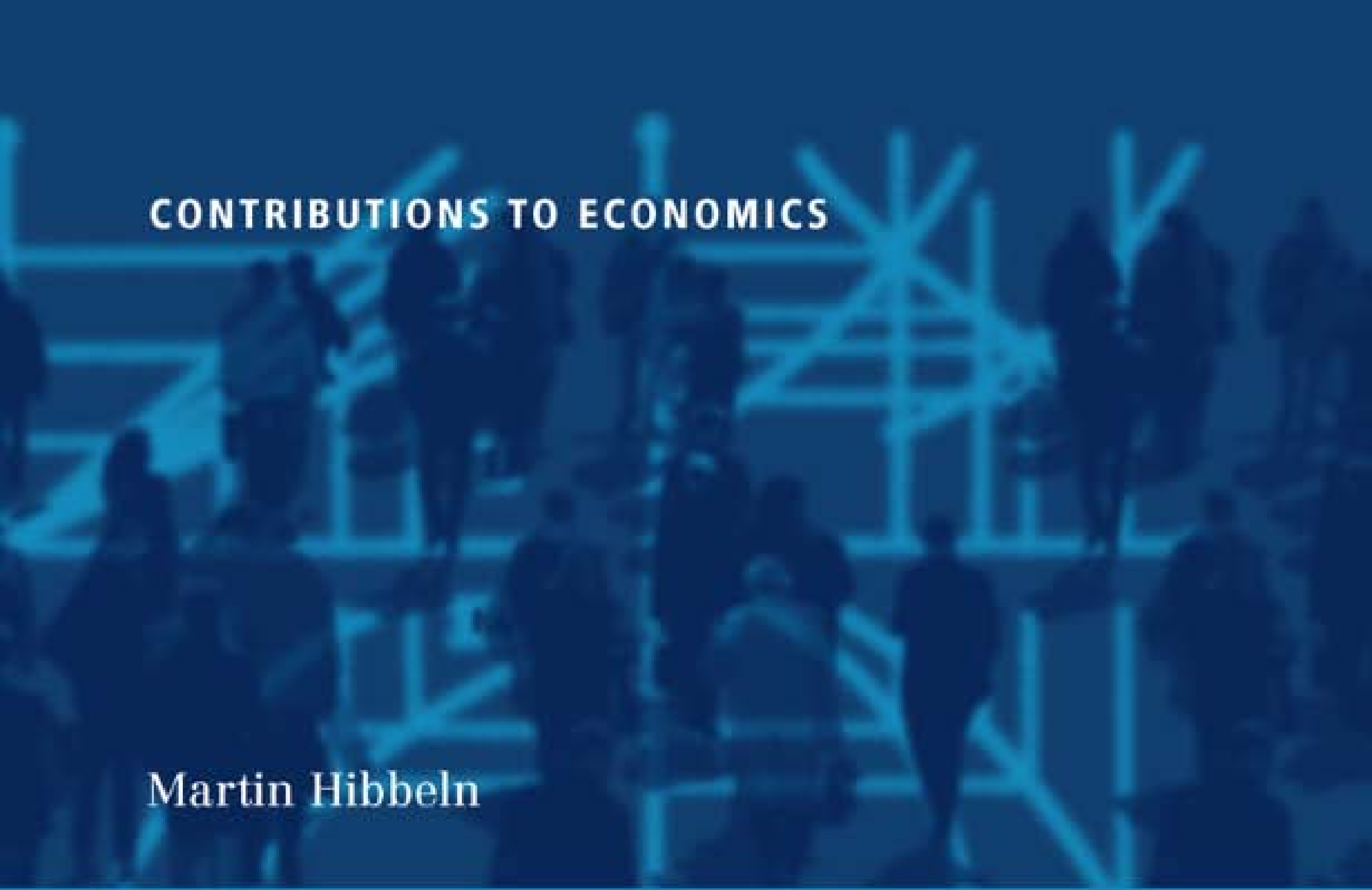

CONTRIBUTIONS TO ECONOMICS

Martin Hibbeln

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Concentration Risk and Basel II

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Risk Management in Credit Portfolios

Concentration Risk and Basel II

Physica-Verlag

Dr. Martin Hibbeln
Technische Universität Braunschweig
Institute of Finance
Carl-Friedrich-Gauß-Faculty
Abt-Jerusalem-Str. 7
38106 Braunschweig
Germany
martin.hibbeln@tu-bs.de

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Foreword

Over the last 10 years, there has hardly been a topic that has occupied the credit sector more than the appropriate determination of the capital backing of credit risk positions. Even after the adoption of the capital requirements by the Basel Committee on Banking Supervision "Basel II" in June 2004, the great relevance of this topic is still present because many types of banking risk are not taken into account. The importance of such risk types is also recognized within the framework of Basel II. According to Pillar 2, "there are three main areas that might be particularly suited to treatment: risks considered under Pillar 1 that are not fully captured by the Pillar 1 process (e.g. credit concentration risk); those factors not taken into account by the Pillar 1 process (e.g. interest rate risk in the banking book, business and strategic risk); and factors external to the bank (e.g. business cycle effects)". In this context especially the consideration of concentration risks is a very important task since concentration risks in mortgage banks can be seen as one relevant cause of the financial crisis.

Against this background, Martin Hibbeln has set himself the targets of analyzing concentration risks in detail and of consistently integrating concentration risks into the Basel II model. First, the author deals with regulatory principles of the European Banking Supervision, which have to be considered in the framework of concentration risk measurement. In addition, he focuses on the question whether or not credit concentrations stemming from bank specialization have a risk increasing effect. The subsequent theoretical analysis takes the Asymptotic Single Risk Factor (ASRF) framework of Gordy as a starting point since this environment underlies the Internal Ratings-Based (IRB) Approach of Basel II. For the purpose of extending this model, Martin Hibbeln addresses two types of concentration risk: name concentrations and sector concentrations. With regard to name concentrations, he determines credit portfolio sizes for different portfolio structures that lead to a violation of the assumptions of the ASRF model. The results are of great practical relevance since on this basis a bank is able to identify concentration risks in their credit portfolios. He also analyzes available granularity adjustments concerning their suitability for the measurement of name concentrations. With respect to sector concentrations, Martin Hibbeln modifies existing approaches to measuring concentration risks in order to consistently extend the Basel II framework. He shows how to implement the approaches for practical application and gives a detailed analysis with regard to measurement accuracy and runtime of the procedures. Again, the results are of practical importance since the analysis shows in detail which procedure shall be implemented. Furthermore, he analyzes the adequacy of the non-coherent risk measure Value-at-Risk (VaR), which is often criticized in the literature. For this purpose, all studies in question are undertaken by the use of the coherent measure Expected Shortfall (ES), as well. Surprisingly, the respective results do not show significant differences and consequently, the use of the VaR seems to be unproblematic when determining risk concentrations.

All in all, this book deals with a relevant topic within the framework of credit risk management. In this context the author succeeds impressively in connecting theoretical results and practical applications, which in turn implies the book to be suitable for academics as well as practitioners. Against this background, I wish this innovative and inventive work the high degree of attention it undoubtedly deserves due to its quality.

Braunschweig, Germany Marc Gürtler
April 2010

Preface

This monograph was written while I was a research associate at the Institute of Finance at the Technische Universität Braunschweig. It was accepted as my doctoral thesis by the Carl-Friedrich-Gauß-Faculty of the Technische Universität Braunschweig in March 2010. At this point I would like to thank all those who have contributed to the success of this work and who have made the time of my dissertation project very exciting and enjoyable.

First of all, my thanks go out to my supervisor, Prof. Dr. Marc Gürtler. Our recurrent technical and non-technical discussions, our sporty competitions as well as the necessary academic freedom I was given were of great importance for the development of this work and for the excellent working atmosphere. Furthermore, I want to thank Prof. Dr. Gernot Sieg for serving as a reviewer for my doctoral thesis and for the quick preparation of the referee report.

I would also like to thank the entire team of the Institute of Finance. During my dissertation project, I could especially benefit from numerous discussions and cooperative publications with Dr. Dirk Heithecker, Dipl.-Math. Oec. Sven Olboeter, and Dipl.-Math. Oec. Clemens Vöhringer. Moreover, I am particularly grateful to Dipl.-Math. Oec. Franziska Becker, Dipl.-Math. Oec. Julia Stolpe, and Dipl.-Math. Oec. Christine Winkelvos for many inspiring and enjoyable discussions about various topics – mostly not explicitly related to this monograph – during my time as a research associate and partly already during my studies. In addition, I want to thank Silvia Nitschke for the perfect organization of the institute. A special thanks goes to Dr. Kathryn Viemann. During my studies and my doctoral work, she was a great sparring partner and always able to encourage and motivate me.

Representative for the team of central risk management of the VW Bank GmbH, I would like to thank Dipl.-Math. Oec. Stefan Ehlers and Dr. Antje Henne of the LGD-team for the excellent collaboration during my 2-years lasting participation in the Basel II project. There I had the opportunity to convince myself that theoretical knowledge about credit risk modeling can easily be put into banking practice.

Many thanks also go to Oliver Bredtmann, M.Sc., Dipl.-Wirt.-Inf. Markus Weinmann, and Anne Gottschall, B.A., for their patience in checking earlier versions of this monograph and improving my writing style in English.

Last but not least, I want to express my thanks to my family, who has always supported and encouraged me, and especially to my parents, Ingeborg and Gert Hibbeln. It was them who never tired of answering me thousands of "How does it work?"- and "Why?"-questions when I was a child, and I am certain that they have a great part in sparking my scientific curiosity, which finally resulted in this doctoral thesis. Thank you for everything!

Braunschweig, Germany Martin Hibbeln
April 2010

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Abbreviations

$\tilde{a}_i$	Standardized log-return of firm i
$\tilde{A}_T$	Asset value at t = T
$\mathcal{B}(n,p)$	Binomial distribution with parameters n and p
$\mathbb{C}$	Set of all complex numbers
$\tilde{D}$	Default event
$\mathbb{E}(\cdot)$	Expectation value
$ES_{\alpha}^{(\infty)}$	ES at confidence level α of a portfolio with infinite granularity
Δl_1	First-order granularity adjustment
Δl_2	Additional term of the second-order granularity adjustment
$\tilde{L}$	Relative loss
$\tilde{L}_{\text{abs}}$	Absolute loss
$\tilde{\tilde{L}}$	Portfolio loss in an accurately adjusted ASRF model
$\mathcal{L}$	Laplace transform
$\mathbb{N}$	Set of all natural numbers
$\mathcal{N}(\mu,\sigma^2)$	Normal distribution with expectation μ and variance σ^2
$O(\cdot)$	Landau symbol
$\bar{p}$	Average probability of default
$p(\cdot)$	Conditional probability of default
Δp	Shift of the survival probability
$\mathbb{P}(\cdot)$	Probability
$q_{\alpha}^{(n)}$	Quantile of a granular portfolio
$q_{\alpha}^{(\infty)}$	Quantile of an infinitely granular portfolio
Δq	Shift of the loss quantile
Δq_{α}	Multi-factor adjustment
$\Delta q_{\alpha}^{\infty}$	Systematic risk adjustment component of the multi-factor adjustment
$\Delta q_{\alpha}^{\text{GA}}$	Granularity adjustment component of the multi-factor adjustment
$\bar{r}_{\text{Intra}}$	Average intra-sector correlation
$\bar{r}_{\text{Inter}}$	Average inter-sector correlation
$\mathbb{R}$	Set of all real numbers

$\mathbb{V}(\cdot)$	Variance
$VaR_\alpha^{(\infty)}$	VaR at confidence level α of a portfolio with infinite granularity
$VaR_\alpha^{(+)}$	Lower VaR at confidence level α
$VaR_\alpha^{(-)}$	Alternative definition of VaR: maximal loss in the best $100 \cdot \alpha\%$ scenarios
$VaR_\alpha^{(\text{int})}$	Interpolated VaR at confidence level α
$\tilde{x}$	Systematic factor
$\tilde{x}_s$	Risk factor of sector s
$\tilde{Y}$	Systematic part of the portfolio loss
$\tilde{z}_k$	Independent risk factors
$\tilde{Z}$	General idiosyncratic component of the portfolio loss
$\mathbb{Z}$	Set of all integers
$1_{\{\}}$	Indicator variable
$\alpha^{(\infty)}$	Quantile of an infinitely granular portfolio
$\bar{\beta}$	Average weighted inter-sector correlation
Γ	Gamma function
$\delta(\cdot)$	Dirac's delta function
$\tilde{\varepsilon}_i$	Idiosyncratic factor of firm i
$\eta_m(\cdot)$	m-th moment about the mean
$\eta_{m,c}$	m-th conditional moment of the portfolio loss about the mean
$\mu_m(\cdot)$	m-th moment about the origin
$\mu_{m,c}$	m-th conditional moment of the portfolio loss about the origin
ξ_i	Idiosyncratic factor of firm i
ρ	Risk measure
$\sqrt{\rho_i}$	Correlation between firm i and the common factor in a one-factor model
$\bar{\rho}_i$	Correlation between obligor i and the systematic risk factor $\tilde{\tilde{x}}$
$\bar{\rho}_s$	Correlation between sector factor $\tilde{x}_s$ and the systematic risk factor
$\rho_{s,t}^{\text{Inter}}$	Correlation between the risk factors of sector s and t
$\rho_{\text{Intra}}^{(\text{Implied})}$	Implicit intra-sector correlation
φ	Standard normal PDF
φ_2	Bivariate normal PDF
Φ	Standard normal CDF
Φ^{-1}	Inverse standard normal CDF
Φ_2	Bivariate normal CDF
μ	Drift rate or expectation value
μ_X	Parameter of the lognormal and logit-normal distribution
B	Liabilities; Beta function; risk bucket
b_i	Factor loading to the systematic factor
b_k	Coefficients of sector factors
CCF	Credit conversion factor
CDF	Cumulative distribution function

c_i	Factor loading to the idiosyncratic factor; correlation parameter in the comparable one-factor model
COMM	Commitments
D	Diversity score
DF	Diversification factor
d_i	Default threshold of obligor *i;* weighting factor in the model of Pykhtin
EAD	Exposure at default
EC^{mf}	Economic capital in a multi-factor model
EL	Expected loss in relative values
EL_{abs}	Expected loss in absolute values
ELGD	Expected LGD
ES	Expected shortfall
ES_α	ES at confidence level α
f	Probability density function
F	Cumulative distribution function
F^{-1}	Inverse cumulative distribution function
G	Gini coefficient
HHI	Herfindahl–Hirschmann index
I_c	Critical number of credits
J	Number of observations in a historical or Monte Carlo simulation
k	Number of defaults
K	Number of independent factors
LGD	Loss given default
$L_{j:J}$	j-th out of *J* elements of the order statistics
M	Maturity; moment generating function
n	Number of credits
N	Number of observations
n^*	Effective number of credits
N_{PD}	Number of PD-classes
OUT	Current outstandings
p	Survival probability ($=1-\alpha$); probability of a direct default in the model of Davis and Lo
PD	(Unconditional) Probability of default
PDF	Probability density function
q	Infection probability in the model of Davis and Lo
q_α	Lower quantile
q^α	Upper quantile
RC^s	Regulatory capital for sector *s*
Res	Residuum
RR	Recovery rate
S	Annual sales; number of Sectors
SLGD	Third moment of the LGD about the mean
T	Point in time
TCE	Tail conditional expectation

TCE_α	Lower TCE at confidence level α
TCE^α	Upper TCE at confidence level α
UL	Unexpected loss
VaR	Value at risk
VaR_α	Lower VaR at confidence level α
VaR^α	Upper VaR at confidence level α
$VLGD$	Variance of the LGD
W	Wiener process
w_i	Exposure weight of credit i in the portfolio
α	Confidence level; parameter of the beta distribution
$\alpha_{i,k}$	Factor weight of obligor i from Cholesky decomposition
$\alpha_{s,k}$	Factor weight of sector s from Cholesky decomposition
β	Target tolerance; parameter of the beta distribution
λ	Fraction of the idiosyncratic risk that stays in the portfolio
σ	Volatility or standard deviation
σ_X	Parameter of the lognormal and logit-normal distribution
τ	Lagrange multiplier

Chapter 1
Introduction

1.1 Problem Definition and Objectives of This Work

"Risk concentrations are arguably the single most important cause of major problems in banks".[1] On the one hand, dealing with concentration risk is important for the survival of individual banks; therefore, banks should be interested in a proper management of risk concentrations on their own. On the other hand, the Basel Committee on Banking Supervision (BCBS) has found that nine out of the thirteen analyzed banking crises were affected by risk concentrations,[2] which shows that this issue is important for the stability of the whole banking system. Consequently, risk concentrations are also crucial from a regulatory perspective and should therefore be considered when establishing regulatory capital standards.

Recently, the "International Convergence of Capital Measurement and Capital Standards – A Revised Framework",[3] better known as "Basel II", has replaced the former capital accord "Basel I". The objective of the new framework is to strengthen the soundness and stability of the international banking system, which shall mainly be achieved by capital requirements that are aligned more closely to the underlying risk. Although Basel II has sometimes been subject to criticism,[4] there is widely consensus that Basel II promotes the adoption of stronger risk management practices by the banking industry and leads to more transparency. The Minimum Capital Requirements are formulated in the so-called Pillar 1 of Basel II. The first pillar is accompanied by the Supervisory Review Process (Pillar 2), which refers to a proper assessment of capital adequacy by banks and a review of this assessment by

[1]BCBS (2005a), § 770.

[2]Cf. BCBS (2004b), p. 66 f.

[3]Cf. BCBS (2004c, 2005a).

[4]One occasionally expressed criticism is the procyclicality of Basel II. This means that in recession the default risk of firms increase and at the same time, due to higher capital requirements for risky credits, the banks have to reduce their investment activities; thus, recessions could be amplified. For a discussion of this aspect, cf. Gordy and Howells (2006).

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supervisors. The market discipline (Pillar 3) is a set of disclosure requirements, which allows market participants to assess information on the capital adequacy.

Until now, most of the literature on Basel II has focused on parameter estimation and the theoretical framework of Pillar 1. Consequently, these concepts are widely known in academics and practice by now. But it is important to notice that some crucial types of risk, like concentration risk, interest rate risk, or liquidity risk, are not considered in the quantitative capital requirements of Pillar 1. Instead, concerning these types of risks, the requirements are only qualitatively formulated under Pillar 2. Fitch Ratings expressed this shortcoming as follows: “While all three Pillars are integral to the effectiveness of Basel II as a regulatory capital framework, it is often Pillar 1 that receives the bulk of public attention, given its direct and explicit impact on bank capital ratios. It is important that financial institutions and market participants also focus on the Pillar 2 objective of managing enterprise risk, including concentration risk, rigorously and comprehensively”.[5]

The existing literature regarding concentration risk in credit portfolios mainly consists of some documents from banking supervisors, empirical studies on the effect of concentration risk on bank performance, and of some proposed models on the measurement of concentration risk, which range from rather simple and heuristic to sophisticated model-based approaches. However, there is hardly any literature which analyzes the impact of credit concentrations on portfolio risk for different portfolio types or answers the practically relevant question, in which cases the influence of concentration risk is rather small so that it should be unproblematic if a bank does not explicitly measure its concentration risk. Furthermore, it would be valuable to know how good the proposed approaches for the measurement of concentration risk do perform in comparison. Moreover, banks are requested by supervisors “to identify, measure, monitor, and control their credit risk concentrations”,[6] but it is not clear how the models on concentration risk can be implemented in a way that they are consistent with the Basel framework. The main objective of this work is to answer these questions. Beyond that, this work tries to integrate economical and regulatory aspects of concentration risk and seeks to provide a systematic way to get familiar with the topic of concentration risk from the basics of credit risk modeling to present research in the measurement and management of credit risk concentrations.

1.2 Course of Investigation

The fundamentals of credit risk measurement and the quantitative framework of Basel II are presented in Chap. 2. At first, the need of banking regulation in general, the development of banking supervision, as well as the concept of Basel II

[5] Hansen et al. (2009).

[6] See BCBS (2005a), § 773.

is presented briefly. In Sect. 2.2, relevant measures of risk in credit portfolios, like the expected loss (EL), the Value at Risk (VaR), and the Expected Shortfall (ES) are introduced. Then, the asset value model of Merton (1974) is described in Sect. 2.3, which builds the basis of the conditional probability of default within the one-factor model of Vasicek (1987) that is derived in Sect. 2.4. Applying this conditional probability, the binomial model of Vasicek (1987) allows determining the loss distribution for homogeneous credit portfolios, which is demonstrated in Sect. 2.5. Next, the Asymptotic Single Risk Factor (ASRF) model of Gordy (2003) is presented in Sect. 2.6. This model allows an easy calculation of the VaR or the ES for heterogeneous portfolios if there is no concentration risk in the portfolio. As a last step, in Sect. 2.7 the conditional probability of default is integrated into the ASRF model, which leads to the core element of the regulatory capital requirement under Pillar 1.

In Chap. 3, risk concentrations in credit portfolios are discussed. Firstly, different types of concentration risk are described. In Sect. 3.2, it is argued that banks often consciously accept concentrations in their portfolios in order to gain higher returns from specialization, but they should have an additional capital buffer to survive economic downturns. The measurement and management of concentration risk, including relevant regulatory requirements and industry best practices, is presented in Sect. 3.3. Then, some simple, heuristic approaches for the measurement of concentration risk are demonstrated and assessed in Sect. 3.4. After that, a review of the literature on model-based approaches for the measurement of concentration risk is presented in Sect. 3.5.

Chap. 4 deals with the measurement of name concentrations. This type of concentration risk occurs if the weight of single credits in the portfolio does not converge to zero; thus, the individual risk component cannot be completely diversified. The main research questions on name concentrations that are considered in this chapter are:

- In which cases are the assumptions of the ASRF framework critical concerning the credit portfolio size?
- In which cases are currently discussed adjustments for the VaR-measurement able to overcome the shortcomings of the ASRF model?

Concerning the first question, it is analyzed how many credits are at least necessary implying the neglect of undiversified individual risk not to be problematic. Since there exist analytical formulas – the so-called granularity adjustment – which approximate these risks, it is further determined in which cases these formulas are able to lead to desired results. Against this background, in Sect. 4.2 the granularity adjustment is presented and in a next step an expansion of the existing formula is derived. Then, the minimum size of a credit portfolio is determined for several parameter combinations, for the case that only the ASRF formula is used and for the case that the granularity adjustment (and its expansion) is applied. The same analyses, which were performed using the risk measure VaR, are carried out for the risk measure ES in Sect. 4.3. The main results of this chapter are subsumed in Sect. 4.4.

After dealing with name concentrations, the focus of Chap. 5 is on sector concentrations. This type of concentration risk can occur if there is more than one systematic risk factor that influences credit defaults. For example, sector concentrations can arise if a relatively high share of a bank's credit exposure is concentrated in a specific industry sector or geographical location. Concerning sector concentrations, the main research questions that are analyzed in this chapter are:

- How can existing approaches for measuring sector concentration risk be modified and adjusted to be consistent with the Basel framework? Is the risk measure Value at Risk problematic when dealing with sector concentration risk?
- Which methods are capable of measuring concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

In order to deal with these questions, in Sect. 5.2 it is initially determined how a multi-factor model can be parameterized to obtain a capital requirement, which is consistent with Basel II. Then, the models of Pykhtin (2004), Cespedes et al. (2006), and Düllmann (2006) are presented and modified, which have been developed to approximate the risk in the presence of sector concentrations. In Sect. 5.3, the accuracy of these models concerning their ability to measure sector concentration risk is compared. In addition to the accuracy of the results, the emphasis is also put on the runtime of the models, since even with up-to-date computer hardware the computation can still take a very long time. Moreover, the simulation study chosen for the comparison is well-suited to analyze in a quite realistic setting whether there are relevant differences if either the risk measure VaR or ES is used. This question is of high practical relevance, since the VaR is often criticized concerning some theoretical shortcomings that are often illustrated in contrived portfolio examples. These shortcomings could be very problematic in the presence of concentration risk, but nevertheless, the VaR is very often applied in practice and in the literature. The results of these analyses are subsumed in Sect. 5.4.

Chapter 2
Credit Risk Measurement in the Context of Basel II

2.1 Banking Supervision and Basel II

During the last decades, there has been a lot of effort spent on improving and extending the regulation of financial institutions. There are several reasons for a regulation of these institutions, which are mostly different from the regulation of other economic sectors. Even if there are some discussions about tendencies of the banking sector to constitute a monopoly as a result of economies of scale and economies of scope, the empirical evidence is rather scarce.[7] A widely accepted argument is that the (unregulated) banking system is unstable. If a bank is threatened by default or the depositors expect a high default risk, this can lead to a bank run, meaning that many depositors could abruptly withdraw their deposits.[8] This behavior is a consequence of the "sequential service constraint", meaning that whether a depositor gets his deposits depends on the position in the waiting queue.[9] The problem is that, as most banks invest the short term deposits in long term projects (term transformation), there is a high risk of illiquidity of the bank, regardless of whether the bank is overindebted or not. Due to incomplete information, depositors of different institutions could also withdraw their deposits, and this domino effect could finally lead to a collapse of the complete banking system. This type of risk is called "systemic risk".[10] Because of the enormous relevance of banks for the complete economy, the state will usually act as a "lender of last resort", especially in the case of big financial institutions ("too big to fail"-phenomenon) instead of accepting a bank's default, which is due to the presence of systemic risk.[11] Against

[7] Cf. Berger et al. (1993, 1999).

[8] Cf. Diamond and Dybvig (1983).

[9] Cf. Greenbaum and Thakor (1995). This is an important difference to securities where the holder is exposed to a price decline instead.

[10] Cf. Saunders (1987) and Hellwig (1995).

[11] The relevance of this phenomenon has been remarkably shown in the ongoing financial crisis. In 2007 and 2008, there have been many examples of bailouts of financial institutions, such as Bear

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this background, the state is interested in a regulation of the financial system in order to reduce the probability of bank runs and the systemic risk.[12]

The first German banking supervision was established in 1931 after the default of the Danatbank during the Great Depression. This event came along with a massive withdrawal of deposits and bank runs. As a consequence of the default of the Herstatt Bank in 1974, about 52,000 private customers lost their money. Furthermore, many US American banks, which had currency contracts with the Herstatt Bank, did not get back their receivables. This event led to several additional regulations, including the extension of deposit guarantees and the large exposure rules. Moreover, as a result of this default, the central-bank Governors of the Group of Ten (G10) countries founded the Basel Committee on Banking Supervision in the end of 1974, which had the objective to close gaps in international supervisory coverage. In 1988 the Committee introduced the Basel Capital Accord (*Basel I*), which led to a major harmonization of international banking regulation and minimum capital requirements for banks.[13] According to Basel I, it is required that banks hold equity equal to 8% of their risk weighted assets, which are calculated as a percentage between 0% (e.g. for OECD banks) and 100% (e.g. for corporates) of the credit exposure. The basic principle behind this requirement is that the minimum capital requirement, which also implies a maximum leverage, leads to an acceptable maximum probability of default for every single bank. Thus, this restriction of risk should lead to a stabilization of the banking system. The problem is that these capital rules are hardly risk-sensitive – for example, an investment grade and a speculative grade corporate bond require the identical capital. As a consequence, banks have an incentive to deal with risky credits, especially if the regulatory capital constraint is binding. This incentive stems from the risk-shifting problem, which is relevant for every indebted institution, but increases with leverage. This problem is already present for projects with identical expected pay-offs but as risky investments usually offer higher expected profits, the incentive of risk-shifting is even higher. In addition, the Basel Capital Accord offered the possibility of "regulatory capital arbitrage", which is a result of the missing risk-sensitivity, too. A bank with a small capital buffer could bundle its low-risk assets in asset backed securities and sell them to investors. After this transaction, the bank still has

---

Stearns, Fannie Mae, Freddie Mac, and AIG in the United States or IKB and Hypo Real Estate in Germany. But an even stronger argument for the "too big to fail"-phenomenon is the default of Lehman Brothers in September 2008. Probably due to the global diversification of their creditors, the bank's default was apparently assessed as no systemic risk. But the subsequent financial turmoil including the almost complete dry up of the interbank lending market shows that this was a material misjudgment of the U.S. government; cf. the German Council of Economic Experts (2008), p. 122. This default clearly demonstrates the relevance of the "too big to fail"-phenomenon and the negative consequences if a big financial institution still fails, especially in an unstable market environment.

[12]For a more detailed discussion of banking regulation see Gup (2000) or Hartmann-Wendels et al. (2007), p. 355 ff.

[13]Cf. Phillips and Johnson (2000), p. 5 ff., Hartmann-Wendels et al. (2007), p. 391 ff., Henking et al. (2006), p. 2 ff., and BCBS (2009a).

almost the same degree of risk but free capital, which could be used to invest in new, risky projects. Thus, it is obvious to see that the minimum capital requirements of Basel I do not effectively reduce the risk-taking behavior of banks.

Against this background, in 1999 the Basel Committee on Banking Supervision (BCBS) published the First Consultative Package on a New Basel Capital Accord (*Basel II*) with a more risk-sensitive framework. Finally, in 2004/05 the Committee presented the outcome of its work under the title "Basel II: International Convergence of Capital Measurement and Capital Standards – A Revised Framework" (BCBS 2004c, 2005a). In this context, it is interesting to notice that it was intended to maintain the overall level of regulatory capital.[14] Thus, the purpose of the new capital rules is indeed to achieve better risk-sensitivity. Basel II is based on "three mutually reinforcing pillars, which together should contribute to safety and soundness in the financial system".[15] *Pillar 1* contains the Minimum Capital Requirements, which mainly refer to an adequate capital basis for credit risk, but operational risk and market risk are considered, too. *Pillar 2* is about the Supervisory Review Process. In contrast to Pillar 1, which contains quantitative and qualitative elements, Pillar 2 contains qualitative requirements only. These refer to a proper assessment of individual risks – beyond the demands of Pillar 1 – and sound internal processes in risk management. Important risk types that are not captured by Pillar 1 are concentration risks, which are the object of investigation during this study, interest rate risks, and liquidity risks. *Pillar 3* shall improve the market discipline through an enhanced disclosure by banks, e.g. about the calculation of capital adequacy and risk assessment. The New Basel Capital Accord has been implemented in the European Union in 2006 via the Capital Requirement Directive (CRD). Subsequently, the member states of the European Union transposed the directive into national law. In Germany, the corresponding regulations are basically the "Solvabilitätsverordnung" (SolvV), which refers to the first and third Pillar of Basel II, some changes in the "Kreditwesengesetz" (KWG) and the "Großkredit- und Millionenkreditverordnung" (GroMiKV), as well as the "Mindestanforderungen an das Risikomanagement" (MaRisk), implementing the demands of the Pillar 2. These regulations came into effect on 01-01-2007.

As this study deals with credit risk management, only this type of risk will be considered in the following. In contrast to Basel I, the minimum capital requirements of Basel II take the probabilities of default of the individual credits into consideration. The concrete quantitative requirements are based on a framework that measures the 99.9%-Value at Risk of a portfolio, which is the loss that will not be exceeded with a probability of at least 99.9%. The banks are free to choose the Standardized Approach or the Internal Ratings-Based (IRB) Approach, which mainly differ concerning the use of external ratings vs. internal estimates of the obligors' creditworthiness. Furthermore, for non-retail obligors the IRB Approach is subdivided into the Foundation IRB Approach and the Advanced IRB Approach.

[14]BCBS (2001b).

[15]BCBS (2001b), p. 2.

While within the Foundation IRB Approach only the probability of default has to be estimated, banks using the Advanced IRB Approach have to estimate additional parameters, such as the Loss Given Default and the Exposure at Default, which are described in the subsequent Sect. 2.2.1.[16] In this context, it should be noticed that the IRB Approach is not only a regulatory set of rules but the underlying framework often serves as a common fundament in banking practice and for ongoing research in credit risk modeling with several improvements and applications.[17] Against this background, it is useful to have a deeper understanding of the concrete credit risk measurement and credit portfolio modeling as a basis of improving the management of credit risk. Thus, in the following there will be a short introduction on individual risk parameters and risk measures in a credit portfolio context, and a detailed explanation of the framework underlying the IRB Approach.

2.2 Measures of Risk in Credit Portfolios

2.2.1 Risk Parameters and Expected Loss

Before the parameters for the quantification of credit risk are explained, we start with some short comments about the general notation. In the following, stochastic variables are marked with a tilde "$\sim$", e.g. $\tilde{x}$ denotes that x is a random variable. Furthermore, "$\mathbb{E}(\tilde{x})$" stands for the expectation value and "$\mathbb{V}(\tilde{x})$" for the variance of the random variable $\tilde{x}$. Similarly, "$\mathbb{P}(\tilde{x} = a)$" denotes the probability that $\tilde{x}$ takes the value a. The random variable $1_{\{\tilde{x} > a\}}$, which is also called an indicator variable, is defined as

$$1_{\{\tilde{x} > a\}} = \begin{cases} 1 & \text{if } \tilde{x} > a, \\ 0 & \text{if } \tilde{x} \leq a. \end{cases} \tag{2.1}$$

Thus, the indicator variable takes the value one if the event specified in brackets occurs, and zero otherwise. Using this notation, the parameters for the quantification of credit risk can be introduced. The potential loss of a credit is usually expressed as a product of three components: The default indicator variable, the loss given default, and the exposure at default.

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[16]Details concerning the concrete regulatory requirements and a comparison of these approaches can be found in Heithecker (2007), especially in Sect. 3.

[17]E.g. the underlying one-factor Gaussian copula model with its implied correlation is market standard for pricing CDOs, cf. Burtschell et al. (2007), p. 2, similar to the model of Black and Scholes for options with its implied volatility. Examples for extensions of the standard Gaussian copula model are Andersen and Sidenius (2005a, b) or Laurent and Gregory (2005). Furthermore, several smaller banks use the regulatory capital formulas for their internal capital adequacy assessment process; cf. BCBS (2009b), p. 14.

Firstly, the default event of an obligor is indicated by the *default indicator variable* $1_{\{\tilde{D}\}}$ that takes the value one if the (uncertain) default event $\tilde{D}$ occurs and zero otherwise.[18] The *probability of default* (PD) of an obligor is defined by $\mathbb{P}\left(1_{\{\tilde{D}\}} = 1\right) =: PD$. In context of the Basel Framework, the PD is the probability that an obligor defaults within 1 year.[19] The Basel Committee on Banking Supervision defines a default as follows: "A default is considered to have occurred with regard to a particular obligor when either or both of the two following events have taken place:

- The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realizing security (if held).
- The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered as being past due once the customer has breached an advised limit or been advised of a limit smaller than current outstandings".[20]

It is important to notice that beside this definition there exist several other definitions of default[21] so that a credit that is defaulted in Bank A could be treated as non-defaulted in Bank B. But as the definition above has to be implemented at least for regulatory purposes, it can be seen as the conjoint definition of default.

Secondly, the *loss given default* (LGD) gives the fraction of a loan's exposure that cannot be recovered by the bank in the event of default. Besides obligor-specific characteristics the LGD can highly depend on contract-specific characteristics such as the value of collateral and the seniority of the credit obligation. The uncertain LGD is denoted by the random variable $\widetilde{LGD}$, whereas the expected LGD is denoted by $\mathbb{E}(\widetilde{LGD}) =: ELGD$. There also exists a direct link between the loss given default and the so-called recovery rate (RR): $\widetilde{RR} = 1 - \widetilde{LGD}$. Both variables usually take values between 0% and 100% but the LGD can also be higher than 100% as workout costs occur when the bank tries to recover (parts of) the outstanding exposure. If the bank fails to recover the loan, the total loss amount can be higher than the defaulted exposure leading to an effective LGD of more than 100% and to a RR of less than 0%, respectively.

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[18] In this study, it is not explicitly differentiated between a default of a single loan or of a firm. In this context, it should be noted that for corporates a defaulting loan is usually associated with a default of the firm; consequently, all other loans of the firm are considered as defaulted, too. Contrary, in retail portfolios the loans are often handled separately; thus, a default of one loan does not imply a default of all other loans of this obligor.

[19] See BCBS (2005a), §§ 285, 331.

[20] BCBS (2005a), § 452. For further details on the definition of default, including a specification of "unlikeliness to pay" see BCBS (2005a), §§ 453–457.

[21] A survey of different definitions of default and their impact on the computed recovery rates can be found in Grunert and Volk (2008).

Thirdly, the *exposure at default* (EAD) consists of the current outstandings (OUT), which are already drawn by the obligor. Furthermore, the obligor could draw a part of the commitments (COMM) leading to an increased EAD. This part is called the credit conversion factor (CCF). Thus, the (uncertain) EAD can be defined as[22]

$$\widetilde{EAD} := OUT + \widetilde{CCF} \cdot COMM \tag{2.2}$$

with $0 \leq \widetilde{CCF} \leq 1$. Despite the fact that the exposure at default is a random variable, it is often associated with "the *expected* gross exposure of the facility upon default of the obligor",[23] that means

$$EAD := OUT + \mathbb{E}\left(\widetilde{CCF}\right) \cdot COMM. \tag{2.3}$$

In this study, the exposure at default is mostly assumed to be deterministic, which leads to identity of the random variable $\widetilde{EAD}$ and the expected value EAD.

Using these three components, we can quantify the loss of a single credit or of a credit portfolio (PF) that consists of n different loans. The loss in absolute values of a single credit $i \in \{1, ..., n\}$ is denoted by $\tilde{L}_{\text{abs},i}$:

$$\tilde{L}_{\text{abs},i} = \widetilde{EAD}_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}. \tag{2.4}$$

Thus, a default of loan i leads to an uncertain loss amount of $\widetilde{EAD}_i \cdot \widetilde{LGD}_i$, which is the fraction LGD of the exposure at default. Similarly, we name the absolute loss of the whole portfolio $\tilde{L}_{\text{abs,PF}}$, which can be calculated as the sum of all individual losses:

$$\tilde{L}_{\text{abs,PF}} = \sum_{i=1}^{n} \tilde{L}_{\text{abs},i} = \sum_{i=1}^{n} \widetilde{EAD}_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}. \tag{2.5}$$

The expected loss $EL_{\text{abs},i}$ of loan i is given by

$$EL_{\text{abs},i} = \mathbb{E}\left(\tilde{L}_{\text{abs},i}\right) = \mathbb{E}\left(\widetilde{EAD}_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right) = EAD_i \cdot ELGD_i \cdot PD_i, \tag{2.6}$$

assuming the random variables to be stochastically independent. The expected loss (EL) is also called "standard risk-costs" and the risk premium contained in the

[22] See Bluhm et al. (2003), p. 24 ff.

[23] BCBS (2005a), § 474.

contractual interest rate should at least include this amount.[24] The expected loss of the whole portfolio $EL_{\text{abs,PF}}$ can be calculated as

$$EL_{\text{abs,PF}} = \sum_{i=1}^{n} EL_{\text{abs},i} = \sum_{i=1}^{n} EAD_i \cdot ELGD_i \cdot PD_i. \tag{2.7}$$

Moreover, we differentiate between the absolute and the relative portfolio loss since it is often useful to write the loss in relative terms in analytical credit risk modeling. The relative portfolio loss results when the absolute loss is divided by the total exposure, and will simply be denoted by $\tilde{L}$ in the following:

$$\tilde{L} = \frac{\tilde{L}_{\text{abs,PF}}}{\sum\limits_{j=1}^{n} \widetilde{EAD}_j} = \sum_{i=1}^{n} \frac{\widetilde{EAD}_i}{\sum\limits_{j=1}^{n} \widetilde{EAD}_j} \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} = \sum_{i=1}^{n} \tilde{w}_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}, \tag{2.8}$$

where $\tilde{w}_i := \widetilde{EAD}_i \big/ \sum\limits_{j=1}^{n} \widetilde{EAD}_j$ is the exposure weight of credit i in the portfolio. Using this notation and assuming deterministic exposure weights $w_i = EAD_i \big/ \sum\limits_{j=1}^{n} EAD_j$, the expected relative portfolio loss can be written as

$$EL = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot PD_i. \tag{2.9}$$

2.2.2 *Value at Risk, Tail Conditional Expectation, and Expected Shortfall*

For an individual loan, the expected loss is the most important risk measure as it significantly influences the contractual interest rate. However, on aggregate portfolio level the quantification of additional risk measures is worthwhile. For instance, it is useful for a bank to get knowledge of the possible portfolio loss in some kind of worst case scenario, which is usually defined with respect to a given confidence level α. Based on this, a bank can determine how much capital is needed to survive such scenarios. There exist several approaches to quantify these capital requirements. Firstly, there are different measures for risk quantification, e.g. the Value at Risk, the Tail Conditional Expectation, and the Expected Shortfall, which will be defined and explained below. Secondly, the capital requirements differ depending on their objective. In Basel II the *regulatory capital* requirement is based on the unexpected loss, which is the difference between the Value at Risk with confidence

[24] Cf. Schroeck (2002), p. 171 f.

level $\alpha = 99.9\%$ and the EL,[25] within a 1-year horizon. Furthermore, banks often internally measure their *economic capital* requirement, which can be defined as the capital level that bank shareholders would choose in absence of capital regulation.[26] The economic capital is usually used for the bank's risk management, the pricing system, the internally defined minimum capital requirement, etc.[27] The internal specification of economic capital can differ from the regulatory capital formula, for instance, regarding the used risk measure, the engine for generating the loss distribution, or the time horizon.[28]

For a definition of the risk measures, a mathematical formulation of *quantiles*, or precisely of the upper quantile q^α and the lower quantile q_α, corresponding to a confidence level α is needed. Given the distribution of a random variable $\tilde{X}$, these quantiles are defined as[29]

$$q_\alpha\left(\tilde{X}\right) := \inf\left\{x \in \mathbb{R} | \mathbb{P}\left[\tilde{X} \leq x\right] \geq \alpha\right\}, \tag{2.10}$$

$$q^\alpha\left(\tilde{X}\right) := \inf\left\{x \in \mathbb{R} | \mathbb{P}\left[\tilde{X} \leq x\right] > \alpha\right\}, \tag{2.11}$$

where $\mathbb{R}$ denotes the set of real numbers. If these definitions are applied to continuous distributions, they lead to the same result. Applied to discrete distributions, the upper quantile can exceed the lower quantile.

The *Value at Risk* (VaR) can be described as "the worst expected loss over a given horizon under normal market conditions at a given confidence level".[30] For an exact formulation, the lower Value at Risk $VaR_\alpha(\tilde{L})$ and the upper Value at Risk $VaR^\alpha(\tilde{L})$ at confidence level α have to be distinguished, which are the quantiles of the loss distribution:[31]

$$VaR_\alpha\left(\tilde{L}\right) := q_\alpha\left(\tilde{L}\right) = \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L} \leq l\right] \geq \alpha\right\}, \tag{2.12}$$

$$VaR^\alpha\left(\tilde{L}\right) := q^\alpha\left(\tilde{L}\right) = \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L} \leq l\right] > \alpha\right\}. \tag{2.13}$$

[25] Sometimes the unexpected loss is defined as $UL = \sqrt{\mathbb{V}\left(\tilde{L}\right)}$ instead; see e.g. Bluhm et al. (2003), p. 28.

[26] See Elizalde and Repullo (2007).

[27] Cf. Jorion (2001), p. 383 ff.

[28] An extensive overview of current practices in economic capital definition and modeling can be found in BCBS (2009b).

[29] Acerbi and Tasche (2002b), p. 1489.

[30] Jorion (2001), p. xxii. The first known use of the Value at Risk is in the late 1980s by the global research at J.P. Morgan but the first widely publicized appearance of the term was 1993 in the report of the Group of Thirty (G-30), which discussed best risk management practices; cf. Jorion (2001), p. 22.

[31] Cf. Acerbi (2004), p. 155. The slightly different notation results from the definition of l as a loss instead of a profit variable.

For continuous distributions, the definitions are identical and with the definition of a distribution function $F_L(l) = \mathbb{P}(\tilde{L} \leq l)$ the VaR can also be written in terms of the inverse distribution function:

$$\begin{aligned} VaR_\alpha(\tilde{L}) &= \inf\{l \in \mathbb{R} | \mathbb{P}[\tilde{L} \leq l] \geq \alpha\} \\ &= l \text{ with } \mathbb{P}[\tilde{L} \leq l] = \alpha \\ &= l \text{ with } F_L(l) = \alpha \\ &= F_L^{-1}(\alpha). \end{aligned} \tag{2.14}$$

For discrete distributions, the term "Value at Risk" will be referred to the lower Value at Risk $VaR_\alpha(\tilde{L})$ in the following, according to Gordy (2003) and Bluhm et al. (2003), if not indicated differently. Using $\mathbb{P}[\tilde{L} \leq l] = 1 - \mathbb{P}[\tilde{L} > l]$, it follows from (2.12) that

$$\begin{aligned} VaR_\alpha(\tilde{L}) &= \inf\{l \in \mathbb{R} | 1 - \mathbb{P}[\tilde{L} > l] \geq \alpha\} \\ &= \inf\{l \in \mathbb{R} | \mathbb{P}[\tilde{L} > l] \leq 1 - \alpha\}. \end{aligned} \tag{2.15}$$

From this definition the description of the VaR as the minimal loss in the worst $100 \cdot (1 - \alpha)\%$ scenarios can best be seen.[32] Obviously, this risk measure refers to a concrete quantile of a distribution but neglects the possible losses that can occur in the worst $100 \cdot (1 - \alpha)\%$ scenarios.

A risk measure that incorporates these low-probable extreme losses, the so-called tail of the distribution, is the *Tail Conditional Expectation* (TCE). Similar to (2.12) and (2.13) the lower Tail Conditional Expectation $TCE_\alpha(\tilde{L})$ and the upper Tail Conditional Expectation $TCE^\alpha(\tilde{L})$ at confidence level α are defined as the conditional expectations above the corresponding α-quantiles:[33]

$$TCE_\alpha(\tilde{L}) := \mathbb{E}(\tilde{L} | \tilde{L} \geq q_\alpha) = \frac{\mathbb{E}\left(\tilde{L} \cdot 1_{\{\tilde{L} \geq q_\alpha\}}\right)}{\mathbb{P}(\tilde{L} \geq q_\alpha)}, \tag{2.16}$$

$$TCE^\alpha(\tilde{L}) := \mathbb{E}(\tilde{L} | \tilde{L} \geq q^\alpha) = \frac{\mathbb{E}\left(\tilde{L} \cdot 1_{\{\tilde{L} \geq q^\alpha\}}\right)}{\mathbb{P}(\tilde{L} \geq q^\alpha)}. \tag{2.17}$$

Consequently, the TCE is always higher than the corresponding VaR at a given confidence level and can differ for discrete distributions according to the definition

[32] Cf. Acerbi (2004), p. 153.

[33] Acerbi and Tasche (2002b), p. 1490. The loss quantiles $q_\alpha(\tilde{L})$ and $q^\alpha(\tilde{L})$ are abbreviated with q_α and q^α, respectively, to achieve a shorter notation.

of the quantile. For continuous distributions, the upper and lower quantiles are identical and therefore both definitions of TCE equal:

$$TCE^{\alpha}_{\text{cont}}(\tilde{L}) = TCE_{\alpha,\text{cont}}(\tilde{L}) = \mathbb{E}(\tilde{L}|\tilde{L} \geq q_\alpha) = \frac{\mathbb{E}\left(\tilde{L} \cdot 1_{\{\tilde{L} \geq q_\alpha\}}\right)}{\mathbb{P}(\tilde{L} \geq q_\alpha)}$$
$$= \frac{1}{1-\alpha}\mathbb{E}\left(\tilde{L} \cdot 1_{\{\tilde{L} \geq q_\alpha\}}\right). \tag{2.18}$$

Acerbi and Tasche (2002b) introduced a similar risk measure, the *Expected Shortfall* (ES):[34]

$$ES_\alpha(\tilde{L}) := \frac{1}{1-\alpha} \cdot \left(\mathbb{E}\left(\tilde{L} \cdot 1_{\{\tilde{L} \geq q_\alpha\}}\right) - q_\alpha \cdot \left(\mathbb{P}(\tilde{L} \geq q_\alpha) - (1-\alpha)\right)\right). \tag{2.19}$$

In contrast to the VaR and the TCE, the ES only depends on the distribution and the confidence level α but not on the definition of the quantile. Looking at the second term, if the probability that $\tilde{L} \geq q_\alpha$ is higher than $(1-\alpha)$, this fraction has to be subtracted from the conditional expectation. If the probability equals $(1-\alpha)$, as for every continuous distribution, the second term vanishes. In this case, the ES is identical to the TCE. An alternative representation of (2.19) is:[35]

$$ES_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_\alpha^1 q^u du. \tag{2.20}$$

The intuition behind the ES and the difference between TCE and ES can be demonstrated with the exemplary probability mass function of a discrete random variable shown in Table 2.1 and the corresponding Fig. 2.1.

In this example, the upper as well as the lower VaR at confidence level $\alpha = 0.95$ is 7%. The corresponding TCE is the expectation conditional on a loss of greater or equal to 7%, which is $7.\bar{4}\%$ in the example. As can be seen in the figure, the probability of the considered events is not equal to 5% but 9%. In contrast to the TCE, for the calculation of the ES, the light grey area is subtracted, which is the

Table 2.1 Loss distribution for an exemplary portfolio

Relative Loss l (in %)	2	4	5	7	8
$\mathbb{P}(\tilde{L} = l)$	80%	10%	1%	5%	4%
$\mathbb{P}(\tilde{L} \leq l)$	80%	90%	91%	96%	100%
$\mathbb{P}(\tilde{L} \geq l)$	100%	20%	10%	9%	4%

[34] Acerbi and Tasche (2002b), p. 1491.

[35] Acerbi and Tasche (2002b), p. 1492.

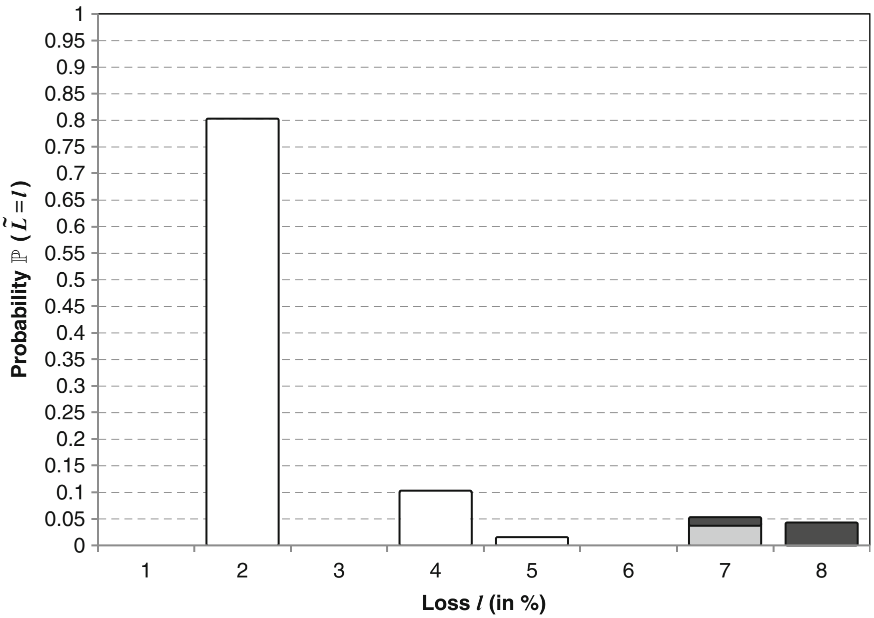

Fig. 2.1 Probability mass function of portfolio losses for an exemplary portfolio

second term of (2.19), and only the dark grey area with a probability of 5% is considered. Thus, the ES is usually higher than the TCE and here we have an ES of 7.8%. Moreover, we can see that the VaR as well as the TCE make a jump if the confidence level is increased from slightly below to slightly above 96%, whereas the ES remains stable because the weight of 7% losses only changes from almost zero to exactly zero.

Subsequently, the calculation of the different risk measures will be demonstrated for the discrete loss distribution of Table 2.1. For this purpose, the confidence levels $\alpha = 0.9$ and $\alpha = 0.95$ are chosen. The upper and lower VaR at these confidence levels are given as

$$VaR_{0.9}(\tilde{L}) = q_{0.9}(\tilde{L}) = \inf\{l \in \mathbb{R} | \mathbb{P}[\tilde{L} \leq l] \geq 0.9\} = 4\,\%,$$

$$VaR^{0.9}(\tilde{L}) = q^{0.9}(\tilde{L}) = \inf\{l \in \mathbb{R} | \mathbb{P}[\tilde{L} \leq l] > 0.9\} = 5\,\%,$$

$$VaR_{0.95}(\tilde{L}) = q_{0.95}(\tilde{L}) = 7\,\%,$$

$$VaR^{0.95}(\tilde{L}) = q^{0.95}(\tilde{L}) = 7\%.$$

It can be seen that the upper and lower VaR are different if there exists a loss outcome l with $\mathbb{P}(l) > 0$ so that $\mathbb{P}[\tilde{L} \leq l] = \alpha$. The same is true for the corresponding TCEs:

$$TCE_{0.9}(\tilde{L}) = \frac{\mathbb{E}\left(\tilde{L}\cdot 1_{\{\tilde{L}\geq q_{0.9}\}}\right)}{\mathbb{P}(\tilde{L}\geq q_{0.9})} = \frac{1}{0.2}(0.1\cdot 4+0.01\cdot 5+0.05\cdot 7+0.04\cdot 8) = 5.6\,\%,$$

$$TCE^{0.9}(\tilde{L}) = \frac{\mathbb{E}\left(\tilde{L}\cdot 1_{\{\tilde{L}\geq q^{0.9}\}}\right)}{\mathbb{P}(\tilde{L}\geq q^{0.9})} = \frac{1}{0.1}(0.01\cdot 5+0.05\cdot 7+0.04\cdot 8) = 7.2\,\%,$$

$$TCE_{0.95}(\tilde{L}) = \frac{1}{0.09}(0.05\cdot 7+0.04\cdot 8) = 7.\bar{4}\,\%,$$

$$TCE^{0.95}(\tilde{L}) = \frac{1}{0.09}(0.05\cdot 7+0.04\cdot 8) = 7.\bar{4}\,\%.$$

According to (2.19), there is only one definition of ES, which results in

$$\begin{aligned} ES_{0.9}(\tilde{L}) &= \frac{1}{1-0.9}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_{0.9}\}}\right] - q_{0.9}\left[\mathbb{P}\left[\tilde{L}\geq q_{0.9}\right]-(1-0.9)\right]\right) \\ &= \frac{1}{1-0.9}([0.1\cdot 4+0.01\cdot 5+0.05\cdot 7+0.04\cdot 8]-4\cdot[0.2-0.1]) = 7.2\,\%, \\ ES_{0.95}(\tilde{L}) &= \frac{1}{1-0.95}([0.05\cdot 7+0.04\cdot 8]-7\cdot[0.09-0.05]) = 7.8\,\%. \end{aligned}$$

For demonstration purposes, an ES-definition based on the upper instead of the lower quantile is calculated, too:

$$\begin{aligned} ES^{0.9}(\tilde{L}) &= \frac{1}{1-0.9}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q^{0.9}\}}\right] - q^{0.9}\left[\mathbb{P}\left[\tilde{L}\geq q^{0.9}\right]-(1-0.9)\right]\right) \\ &= \frac{1}{1-0.9}([0.01\cdot 5+0.05\cdot 7+0.04\cdot 8]-5\cdot[0.1-0.1]) = 7.2\,\%, \\ ES^{0.95}(\tilde{L}) &= \frac{1}{1-0.95}([0.05\cdot 7+0.04\cdot 8]-7\cdot[0.09-0.05]) = 7.8\,\%. \end{aligned}$$

It can be seen that the definitions based on the upper as well as on the lower quantile lead to the same result, even if the calculation itself differs for $\alpha = 0.9$.

2.2.3 *Coherency of Risk Measures*

As demonstrated in Sect. 2.2.2, there exist several measures that could be used for quantifying credit portfolio risk. To identify suitable risk measures, it is reasonable to analyze which mathematical properties should be satisfied by a risk measure to correspond with rational decision making. Based on this, it is possible to evaluate different measures concerning their ability to measure risk in the desired way. Against this background, Artzner et al. (1997, 1999) define a set of four axioms and call the risk measures which satisfy these axioms "coherent". Some authors even mention that these axioms are the minimum requirements which must be fulfilled by a risk measure and therefore do not distinguish between coherent and non-coherent risk measures but denominate only measures that satisfy these axioms "risk measures".[36]

For a mathematical description of these properties, it is assumed that $\mathcal{G}$ is as set of real-valued random variables (for instance the losses of a set of credits). A function $\rho : \mathcal{G} \to \mathbb{R}$ is called a *coherent risk measure* if the following axioms are satisfied:[37]

(A) *Monotonicity*: $\forall \tilde{L}_1, \tilde{L}_2 \in \mathcal{G}$ with $\tilde{L}_1 \leq \tilde{L}_2 \Rightarrow \rho(\tilde{L}_1) \leq \rho(\tilde{L}_2)$.

This means that if the losses of portfolio 1 are smaller than the losses of portfolio 2, then the risk of portfolio 1 is smaller than the risk of portfolio 2.

(B) *Subadditivity*: $\forall \tilde{L}_1, \tilde{L}_2 \in \mathcal{G} \Rightarrow \rho(\tilde{L}_1 + \tilde{L}_2) \leq \rho(\tilde{L}_1) + \rho(\tilde{L}_2)$.

This axiom reflects the positive effect of diversification. If two portfolios are aggregated, the combined risk should not be higher than the sum of the individual risks. This also means that a merger does not create extra risk. If this axiom is not fulfilled, there is an incentive to reduce the measured risk by asset stripping. Another positive effect is the enabling of a decentralized risk management. If the risk measure ρ is interpreted as the amount of economic capital that is required as a cushion against the portfolio loss, each division of an institution could measure its own risk and could have access to a specified amount of economic capital because the sum of the measured risk or required capital is an upper barrier of the aggregated risk or required capital.

(C) *Positive homogeneity*: $\forall \tilde{L} \in \mathcal{G}, \forall h \in \mathbb{R}^+ \Rightarrow \rho(h \cdot \tilde{L}) = h \cdot \rho(\tilde{L})$.[38]

If a multiple h of an amount is invested into a position, the resulting loss and the required economic capital will be a multiple h of the original loss, too. It is important to notice that this axiom is not necessarily valid for liquidity risk.[39]

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[36]See e.g. Szegö (2002), p. 1260, and Acerbi and Tasche (2002a), p. 380 f.

[37]Cf. Artzner et al. (1999), p. 209 ff. The definition of the axioms is slightly different from the original set because here the variables $\tilde{L}_1, \tilde{L}_2$ correspond to a portfolio loss instead of a future net worth of a position; see also Bluhm et al. (2003), p. 166. Moreover, it has to be noted that within the axioms of coherency the loss variables $\tilde{L}, \tilde{L}_i$ refer to absolute instead of relative losses.

[38]$\mathbb{R}^+$ denotes all real numbers greater than zero.

[39]The liquidity risk argument is: "If I double an illiquid portfolio, the risk becomes more than double as much!"; see Acerbi and Scandolo (2008), p. 3. Therefore, axiom (B) and (C) are sometimes replaced by a single weaker requirement of *convexity*: $\forall \tilde{L}_1, \tilde{L}_2 \in \mathcal{G}, \forall h \in [0,1] \Rightarrow \rho\left(h \cdot \tilde{L}_1 + (1-h) \cdot \tilde{L}_2\right) \leq h \cdot \rho\left(\tilde{L}_1\right) + (1-h) \cdot \rho\left(\tilde{L}_2\right)$; cf. Carr et al. (2001), Frittelli and Rosazza Gianin (2002) or Föllmer and Schied (2002). Acerbi and Scandolo (2008) agree with the statement above but they deny that the coherency axioms are contradicted by this. They argue that the axiom has to be interpreted in terms of portfolio values and not of portfolios. In *liquid* markets the relationship between a portfolio and the value is linear ("if I double the portfolio I double the value"), and therefore there is no difference whether thinking about portfolios or portfolio values. However, in *illiquid* markets the value function is usually non-linear. Based on a proposal of a

(D) *Translation invariance*: $\forall \tilde{L} \in \mathcal{G}, \forall m \in \mathbb{R} \Rightarrow \rho(\tilde{L} + m) = \rho(\tilde{L}) + m.$

If there is an amount m in the portfolio that is lost at the considered horizon with certainty, then the risk is exactly this amount higher than without this position.

In the following, it will be shown that the VaR is not a coherent risk measure as it lacks of subadditivity. The same is true for the TCE if the distribution is discrete.[40] However, the ES satisfies all four axioms and therefore is a (coherent) risk measure.

The *monotonicity* of the VaR directly follows from its definition. If a stochastic variable $\tilde{\varepsilon} \geq 0$ is introduced so that $\tilde{L}_1 + \tilde{\varepsilon} = \tilde{L}_2$, it follows that

$$\begin{aligned} VaR_\alpha\left(\tilde{L}_1\right) &= \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L}_1 \leq l\right] \geq \alpha\right\} \\ &\leq \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L}_1 \leq l - \tilde{\varepsilon}\right] \geq \alpha\right\} \\ &= \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L}_2 \leq l\right] \geq \alpha\right\} \\ &= VaR_\alpha\left(\tilde{L}_2\right). \end{aligned} \tag{2.21}$$

To show the *positive homogeneity*, a variable $l = h \cdot x$ is introduced so that it follows $\forall \tilde{L} \in \mathcal{G}$ and $\forall h \in \mathbb{R}^+$:

$$\begin{aligned} VaR_\alpha\left(h \cdot \tilde{L}\right) &= \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[h \cdot \tilde{L} \leq l\right] \geq \alpha\right\} \\ &= h \cdot \inf\left\{x \in \mathbb{R} | \mathbb{P}\left[h \cdot \tilde{L} \leq h \cdot x\right] \geq \alpha\right\} \\ &= h \cdot VaR_\alpha\left(\tilde{L}\right). \end{aligned} \tag{2.22}$$

Furthermore, the VaR is *translation invariant* since $\forall \tilde{L} \in \mathcal{G}$ and with $l = x + m$ we obtain:

$$\begin{aligned} VaR_\alpha\left(\tilde{L} + m\right) &= \inf\left\{l \in \mathbb{R} | \mathbb{P}\left[\tilde{L} + m \leq l\right] \geq \alpha\right\} \\ &= \inf\left\{x \in \mathbb{R} | \mathbb{P}\left[\tilde{L} + m \leq x + m\right] \geq \alpha\right\} + m \\ &= VaR_\alpha\left(\tilde{L}\right) + m. \end{aligned} \tag{2.23}$$

The lack of *subadditivity* of the VaR is sufficient to be shown by an example. It is assumed that a loan A and a loan B both have a PD of 6%, an LGD of 100%, and an EAD of 0.5. The VaR at confidence level 90% of each loan is

$$VaR_{0.9}\left(\tilde{L}_A\right) = VaR_{0.9}\left(\tilde{L}_B\right) = 0. \tag{2.24}$$

formalism for liquidity risk and a proposed non-linear value function, the authors show that liquidity risk is compatible with the axioms of coherency. Further they show that convexity is not a new axiom but a result of the other axioms under their formalism.

[40] Cf. Acerbi and Tasche (2002b), p. 1499, for an example. As the rest of the study focuses on the VaR and the ES, only these risk measures will be analyzed regarding coherency.

If both loans are aggregated into a portfolio, the risk should be smaller or equal to the sum of the individual risks. Assuming that the default events are independent of each other, the probability distribution is given as

$$\begin{aligned} \mathbb{P}\left(\tilde{L}_A+\tilde{L}_B=0\right) &= (1-0.06)^2 = 88.36\%,\\ \mathbb{P}\left(\tilde{L}_A+\tilde{L}_B=0.5\right) &= 0.06\cdot(1-0.06)+(1-0.06)\cdot 0.06 = 11.28\%,\\ \mathbb{P}\left(\tilde{L}_A+\tilde{L}_B=1\right) &= 0.06^2 = 0.36\%. \end{aligned} \tag{2.25}$$

Thus, the VaR at confidence level 90% of the portfolio is

$$VaR_{0.9}\left(\tilde{L}_A+\tilde{L}_B\right) = 0.5 \tag{2.26}$$

leading to

$$VaR_{0.9}\left(\tilde{L}_A+\tilde{L}_B\right) > VaR_{0.9}\left(\tilde{L}_A\right) + VaR_{0.9}\left(\tilde{L}_B\right). \tag{2.27}$$

This shows that the VaR can be superadditive and thus it is not a coherent risk measure. An important exception is the class of elliptical distributions, e.g. the multivariate normal distribution and the multivariate student's t-distribution, for which the VaR is indeed coherent.[41] As credit risk usually cannot be sufficiently described by elliptical distributions, the lack of coherency can be very critical.

To demonstrate the coherency of ES, it is helpful to use a further representation of (2.19). The purpose is to integrate the second term of (2.19) into the expectation of the first term. Defining a variable $1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}$ that is

$$1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}} := \begin{cases} 1_{\{\tilde{L}\geq q_\alpha\}} & \text{if } \mathbb{P}\left[\tilde{L}=q_\alpha\right]=0,\\ 1_{\{\tilde{L}\geq q_\alpha\}} - \frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}\cdot 1_{\{\tilde{L}=q_\alpha\}} & \text{if } \mathbb{P}\left[\tilde{L}=q_\alpha\right]>0, \end{cases} \tag{2.28}$$

the ES can be written as[42]

$$ES_\alpha\left(\tilde{L}\right) = \frac{1}{1-\alpha}\cdot\mathbb{E}\left[\tilde{L}\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right]. \tag{2.29}$$

For the proof of coherency the following properties will be used:[43]

$$\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right] = 1-\alpha, \tag{2.30}$$

---

[41] Cf. Embrechts et al. (2002). An interesting result is that under the standard assumption of normally distributed returns, Markowitz μ–σ-efficient portfolios are also μ–VaR-efficient.

[42] Cf. Acerbi et al. (2001), p. 8, and Acerbi and Tasche (2002b), p. 1493. For a formal proof see Appendix 2.8.1.

[43] These properties are derived in Appendix 2.8.1, too. See also Acerbi et al. (2001) for a proof based on the ES-definition using upper instead of lower quantiles.

$$1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}} \in [0,\ 1]. \tag{2.31}$$

From definition (2.28) and property (2.31) it can be seen that the variable $1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}$ is not the "normal" indicator function but can also take values between zero and one. Subsequently, the coherency of ES will be shown.[44] The *monotonicity* of the ES can easiest be shown with the integral representation (2.20). It has already been shown that $q_\alpha(\tilde{L}_1) \leq q_\alpha(\tilde{L}_2)$ for $\tilde{L}_1 \leq \tilde{L}_2$ and it can be seen from (2.21) that the same is true for $q^\alpha(\tilde{L}_1) \leq q^\alpha(\tilde{L}_2)$. Therefore, it follows

$$\begin{aligned} ES_\alpha(\tilde{L}_1) &= \frac{1}{1-\alpha}\int_\alpha^1 q^u(\tilde{L}_1)\,du \\ &\leq \frac{1}{1-\alpha}\int_\alpha^1 q^u(\tilde{L}_2)\,du = ES_\alpha(\tilde{L}_2). \end{aligned} \tag{2.32}$$

Using the positive homogeneity of the quantile from (2.22), the ES can shown to be *positive homogeneous* as well:

$$\begin{aligned} ES_\alpha(h\cdot\tilde{L}) &= \frac{1}{1-\alpha}\cdot\mathbb{E}\left[h\cdot\tilde{L}\cdot 1^{\alpha}_{\{h\cdot\tilde{L}\geq q_\alpha(h\cdot\tilde{L})\}}\right] \\ &= \frac{1}{1-\alpha}\cdot\mathbb{E}\left[h\cdot\tilde{L}\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\right] \\ &= h\cdot ES_\alpha(\tilde{L}). \end{aligned} \tag{2.33}$$

The *translation invariance* can be obtained using $\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right] = 1-\alpha$ (see (2.30)):

$$\begin{aligned} ES_\alpha(\tilde{L}+m) &= \frac{1}{1-\alpha}\cdot\mathbb{E}\left[(\tilde{L}+m)\cdot 1^{\alpha}_{\{(\tilde{L}+m)\geq q_\alpha(\tilde{L}+m)\}}\right] \\ &= \frac{1}{1-\alpha}\cdot\mathbb{E}\left[(\tilde{L}+m)\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\right] \\ &= \frac{1}{1-\alpha}\cdot\mathbb{E}\left[\tilde{L}\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\right] + \frac{m}{1-\alpha}\cdot\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\right] \\ &= ES_\alpha(\tilde{L}) + m. \end{aligned} \tag{2.34}$$

[44] See also Acerbi and Tasche (2002b).

It remains to show the *subadditivity* of the ES. Introducing the random variables $\tilde{L}_1, \tilde{L}_2$ and $\tilde{L}_3 = \tilde{L}_1 + \tilde{L}_2$, the following statement has to be true:

$$ES_\alpha(\tilde{L}_1) + ES_\alpha(\tilde{L}_2) - ES_\alpha(\tilde{L}_3) \geq 0. \tag{2.35}$$

Using representation (2.29) and multiplying by $(1-\alpha)$ leads to

$$\begin{aligned}
&\mathbb{E}\left[\tilde{L}_1 \cdot 1^\alpha_{\{\tilde{L}_1 \geq q_\alpha(\tilde{L}_1)\}} + \tilde{L}_2 \cdot 1^\alpha_{\{\tilde{L}_2 \geq q_\alpha(\tilde{L}_2)\}} - \tilde{L}_3 \cdot 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right] \\
&\quad = \mathbb{E}\left[\tilde{L}_1 \cdot \left(1^\alpha_{\{\tilde{L}_1 \geq q_\alpha(\tilde{L}_1)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) + \tilde{L}_2 \cdot \left(1^\alpha_{\{\tilde{L}_2 \geq q_\alpha(\tilde{L}_2)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right)\right].
\end{aligned} \tag{2.36}$$

If the terms in brackets are analyzed, we find that

$$1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}} = \begin{cases} 1 - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}} \geq 0 & \text{if} \quad \tilde{L}_i > q_\alpha(\tilde{L}_i), \\ 0 - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}} \leq 0 & \text{if} \quad \tilde{L}_i < q_\alpha(\tilde{L}_i), \end{cases} \tag{2.37}$$

with $i \in [1,\ 2]$, due to the fact that $1^\alpha_{\{\tilde{L} \geq q_\alpha\}} \in [0,\ 1]$. Consequently, we have

$$\begin{aligned}
&\tilde{L}_i \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \\
&\quad \geq q_\alpha(\tilde{L}_i) \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \quad \text{if} \quad \tilde{L}_i > q_\alpha(\tilde{L}_i), \\
&\tilde{L}_i \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \\
&\quad \geq q_\alpha(\tilde{L}_i) \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \quad \text{if} \quad \tilde{L}_i < q_\alpha(\tilde{L}_i), \\
&\tilde{L}_i \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \\
&\quad = q_\alpha(\tilde{L}_i) \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \quad \text{if} \quad \tilde{L}_i = q_\alpha(\tilde{L}_i),
\end{aligned} \tag{2.38}$$

and therefore

$$\tilde{L}_i \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right) \geq q_\alpha(\tilde{L}_i) \cdot \left(1^\alpha_{\{\tilde{L}_i \geq q_\alpha(\tilde{L}_i)\}} - 1^\alpha_{\{\tilde{L}_3 \geq q_\alpha(\tilde{L}_3)\}}\right). \tag{2.39}$$

Using this inequality and again $\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right] = 1-\alpha$ according to (2.30), we find that

$$\begin{aligned}
&\mathbb{E}\left[\tilde{L}_1\cdot\left(1^{\alpha}_{\{\tilde{L}_1\geq q_\alpha(\tilde{L}_1)\}}-1^{\alpha}_{\{\tilde{L}_3\geq q_\alpha(\tilde{L}_3)\}}\right)+\tilde{L}_2\cdot\left(1^{\alpha}_{\{\tilde{L}_2\geq q_\alpha(\tilde{L}_2)\}}-1^{\alpha}_{\{\tilde{L}_3\geq q_\alpha(\tilde{L}_3)\}}\right)\right]\\
&\quad\geq q_\alpha(\tilde{L}_1)\cdot\left(\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}_1\geq q_\alpha(\tilde{L}_1)\}}\right]-\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}_3\geq q_\alpha(\tilde{L}_3)\}}\right]\right)\\
&\quad+q_\alpha(\tilde{L}_2)\cdot\left(\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}_2\geq q_\alpha(\tilde{L}_2)\}}\right]-\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}_3\geq q_\alpha(\tilde{L}_3)\}}\right]\right)\\
&=q_\alpha(\tilde{L}_1)\cdot((1-\alpha)-(1-\alpha))+q_\alpha(\tilde{L}_2)\cdot((1-\alpha)-(1-\alpha))\\
&=0.
\end{aligned}\tag{2.40}$$

Thus, in contrast to the VaR, the ES is subadditive. Since all four axioms are fulfilled, the ES is indeed a coherent risk measure. In addition to the ES, there exist several other coherent risk measures. A class of coherent risk measures is given by the so-called spectral measures of risk with the ES as a special case. This class allows defining a risk-aversion function which leads to different coherent risk measures provided that the risk-aversion function satisfies some conditions presented by Acerbi (2002).[45] However, for the rest of this study the focus will be on the (non-coherent) VaR and the (coherent) ES.

2.2.4 Estimation and Statistical Errors of VaR and ES

Only in minor cases the VaR and the ES will directly be calculated by (2.15) and (2.19), respectively. In real-world applications, the risk measures will mostly be computed via historical simulation or Monte Carlo simulation. In a *historical simulation*, the probability distribution of the loss variable or of several risk factors is assumed to be identical to the empirical distribution of a defined period. Moreover, it is assumed that the realizations are independent of each other. For example, future scenarios will be generated by drawing from $J = 52$ historically observed weekly returns with identical probability. In a *Monte Carlo simulation*, there exists an analytic description of the risk drivers and the dependency between risk drivers and portfolio loss but there is no well-known closed form solution of the probability distribution of the portfolio loss. Thus, a large number J of scenarios can be generated by drawing J independent outcomes of the risk drivers. Using the known dependence structure, J outcomes of the portfolio loss can be computed, which build the simulation-based probability distribution of the portfolio loss.

[45]See also Acerbi (2004), p. 168 ff.

This simulation-based distribution converges towards the exact portfolio distribution as $J \to \infty$.

For a historical simulation as well as for a Monte Carlo simulation, the result is given as a sequence $\{L_j\}_{j=1,\ldots,J}$, where each L_j is a realization of the portfolio loss variable $\tilde{L}$. Based on this, the *empirical distribution* is defined as[46]

$$F^{(J)}(l) = \mathbb{P}\left[\tilde{L} \le l\right] = \frac{1}{J} \cdot \sum_{j=1}^{J} 1_{\{L_j \le l\}}. \tag{2.41}$$

For computation of the corresponding VaR and ES, it is useful to introduce the so-called *order statistics* $\{L_{j:J}\}_{j=1,\ldots,J}$. Therefore, the sample is sorted into an increasing order such that

$$L_{1:J} \le L_{2:J} \le \ldots \le L_{J:J}. \tag{2.42}$$

Now, let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor function and the ceiling function of a real number $x \in \mathbb{R}$:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \le x\}, \tag{2.43}$$

$$\lceil x \rceil = \min\{n \in \mathbb{Z} | n \ge x\}, \tag{2.44}$$

where $\mathbb{Z}$ denotes the set of all integers. Then, using the definition of the lower VaR (2.12) and the upper VaR (2.13), the *empirical estimator of VaR* is given as[47]

$$\begin{array}{ll} VaR_{\alpha}^{(J)}\left(\tilde{L}\right) = VaR^{\alpha(J)}\left(\tilde{L}\right) = L_{\lceil J\cdot\alpha\rceil:J} & \text{if} \quad J \cdot \alpha \notin \mathbb{Z}, \\ \left.\begin{array}{l} VaR_{\alpha}^{(J)}\left(\tilde{L}\right) = L_{J\cdot\alpha:J} \\ VaR^{\alpha(J)}\left(\tilde{L}\right) = L_{J\cdot\alpha+1:J} \end{array}\right\} & \text{if} \quad J \cdot \alpha \in \mathbb{Z}. \end{array} \tag{2.45}$$

This means that except for special cases the VaR is simply given by the $J \cdot \alpha$-th element (rounded up) of the ordered loss sequence. An important characteristic of the empirical estimator is its consistency for large J if the lower VaR equals the upper VaR:

$$\lim_{J\to\infty} VaR_{\alpha}^{(J)}\left(\tilde{L}\right) = VaR_{\alpha}\left(\tilde{L}\right) = VaR^{\alpha}\left(\tilde{L}\right). \tag{2.46}$$

Otherwise the empirical estimators of VaR "flip between the possible values $VaR_{\alpha}(\tilde{L})$ and $VaR^{\alpha}(\tilde{L})$".[48]

[46]Cf. Acerbi (2004), p. 166.

[47]Cf. also Acerbi (2004), p. 167.

[48]Acerbi (2004), p. 168. This can be illustrated by the "head-or-tail"-example of Acerbi (2004). Let both equiprobable events be related to the loss of $\{-1, 0\}$. The VaRs are given as $VaR_{0.5} = -1$ and $VaR^{0.5} = 0$ but even for large J the 50%-quantile neither converges to –1 nor to 0 but flips between these values.

The *empirical estimator of ES* can be determined with[49]

$$ES_\alpha^{(J)}\left(\tilde{L}\right) = \frac{1}{J\cdot(1-\alpha)}\cdot\left(\sum_{j=\lceil J\cdot\alpha\rceil}^{J} L_{j:J} - (J\cdot\alpha - \lfloor J\cdot\alpha\rfloor)\cdot L_{\lceil J\cdot\alpha\rceil:J}\right). \tag{2.47}$$

In the example of $J = 52$ weekly returns, the 90%-ES can be computed as

$$ES_\alpha^{(52)}\left(\tilde{L}\right) = \frac{1}{5.2}\cdot\left(\sum_{j=47}^{52} L_{j:52} - (46.8 - 46)\cdot L_{47:52}\right). \tag{2.48}$$

This shows that the ES can be interpreted as the average loss in the worst 5.2 scenarios. As can be seen from (2.47), the last term is negligible if J is large. Thus, for historical simulation with a relatively small number of scenarios it is important to consider this term whereas it could be neglected in Monte Carlo simulations since there is typically a very large number of generated scenarios. When $J\cdot\alpha \in \mathbb{Z}$, the empirical estimator simplifies to

$$ES_\alpha^{(J)}\left(\tilde{L}\right) = \frac{1}{J\cdot(1-\alpha)}\cdot\sum_{j=J\cdot\alpha+1}^{J} L_{j:J}. \tag{2.49}$$

Acerbi and Tasche (2002b) showed that the estimator for the ES is consistent for large J:

$$\lim_{J\to\infty} ES_\alpha^{(J)}\left(\tilde{L}\right) = ES_\alpha\left(\tilde{L}\right). \tag{2.50}$$

As shown in the previous sections, the ES has some significant theoretical advantages in comparison with the VaR. But from a practical perspective, the ES is often criticized to be much less robust than the VaR. Consequently, the theoretical advantages of ES could be useless if the number of observations was limited, and thus the VaR would be a much more reliable risk measure than the ES. The standard argument is reproduced by Acerbi (2004) as follows: “VaR does not even try to estimate the leftmost tail events, it simply neglects them altogether, and therefore it is not affected by the statistical uncertainty of rare events. ES on the contrary, being a function of rare events also, has a much larger statistical error”. Against this background, Acerbi (2004) analyzes the statistical errors of VaR

[49] Cf. Acerbi (2004), p. 166 f.

and ES. For continuous distributions of a random variable $\tilde{X}$, the *variances of the estimators* for large J are given as[50]

$$\mathbb{V}\left(VaR_{\alpha}^{(J)}(\tilde{X})\right) \overset{J \gg 1}{=} \frac{1}{J} \cdot \frac{\alpha \cdot (1-\alpha)}{f(F^{-1}(\alpha))^2}, \tag{2.51}$$

$$\mathbb{V}\left(ES_{\alpha}^{(J)}(\tilde{X})\right) \overset{J \gg 1}{=} \frac{1}{J \cdot (1-\alpha)^2} \cdot \int_{y=0}^{F^{-1}(\alpha)} \int_{z=0}^{F^{-1}(\alpha)} \min(F(y), F(z) - F(y) \cdot F(z)) dz\, dy, \tag{2.52}$$

where F denotes the cumulative distribution function (CDF) of $\tilde{X}$, F^{-1} is the inverse CDF, and $f = dF/dx$ stands for the probability density function (PDF). From (2.51) and (2.52), it can be seen that the estimator of VaR as well as the estimator of ES have the same dependence on the number of trials J. For both estimators, the precision in terms of standard deviation of the demanded statistics can be improved by factor m if the number of trials is increased by factor m^2. However, even if the standard deviations of the estimators are in both cases of order $O(1/\sqrt{J})$,[51] the constant factors could be very different. Therefore, Acerbi (2004) compares the relative error of VaR and of ES for several heavy-tailed probability distributions and confidence levels.[52] He finds that in most cases the relative errors of VaR and ES are very similar. Only in some cases the relative error of ES is at most twice as much as the error of VaR at very high confidence levels. Even if the results of this analysis need not to be true in general, VaR and ES seem to have similar statistical errors and therefore there is no practical burden in implementing the ES instead of the VaR.

2.3 The Unconditional Probability of Default Within the Asset Value Model of Merton

In order to measure the risk of a credit portfolio according to (2.8), it is necessary to specify the stochastic dependence of loan defaults. A widely-used model is the Vasicek model,[53] which is based on the *asset value model* of Merton (1974). In this type of model it is assumed that a firm does not default as a consequence of insufficient liquidity at the moment of repaying a credit because the firm could sell a

[50] Cf. Acerbi (2004), p. 200 f.

[51] The Landau symbol $O(\cdot)$ is defined as in Billingsley (1995), p. 540, A18.

[52] The analyses are performed for lognormal distributions with different volatility parameters and for power law distributions with different shape parameters.

[53] See e.g. Vasicek (1987, 1991, 2002) and Finger (1999, 2001).

part of its assets or it could issue stocks or bonds in order to repay the credit. This can be done as long as the value of liabilities is higher than the value of assets because thenceforward the market participants will not be willing to pay for a security of the firm. Thus, it is assumed that a firm defaults if the asset value $\tilde{A}_T$ is lower than the value of liabilities B payable at time T: $\tilde{A}_T < B$.[54] Consequently, the probability of default is given by

$$PD = \mathbb{P}\left(\tilde{A}_T < B\right). \tag{2.53}$$

The asset value A is modeled as a geometric Brownian motion:[55]

$$dA_t = \mu A_t\, dt + \sigma A_t\, dW_t \quad \text{with} \quad dW_t = \tilde{\varepsilon}\sqrt{dt},\ \tilde{\varepsilon} \sim \mathcal{N}(0,1), \tag{2.54}$$

using the drift rate μ, the volatility σ and the standard Wiener process dW_t.[56] In order to get a closed form solution of the distribution of the asset value at time T, Itô's Lemma is applied to (2.54) leading to[57]

$$dY_t = d\ln A_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW_t. \tag{2.55}$$

This shows that the logarithm of the asset value follows a generalized Wiener process with drift rate $\mu - 1/2\sigma^2$ and variance rate σ^2. As the logarithm of the asset value is normally distributed, the asset value is lognormally distributed. The distribution of the asset value at time T results by integration of (2.55) from $t = 0$ to $t = T$:

$$\begin{aligned}
\ln\left(\frac{\tilde{A}_T}{A_0}\right) &= \ln\tilde{A}_T - \ln A_0 = \int_{t=0}^{T} d\ln A_t \\
&= \int_{t=0}^{T}\left(\mu - \frac{1}{2}\sigma^2\right)dt + \int_{t=0}^{T}\sigma dW_t \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\left(\tilde{W}_T - W_0\right) \\
&\Leftrightarrow \tilde{A}_T = A_0 \cdot \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{W}_T\right],
\end{aligned} \tag{2.56}$$

[54]As can be seen by this expression, the liabilities are assumed to have the structure of a zero coupon bond that has to be paid completely at time T.

[55]A normal distribution with expectation μ and variance σ^2 is indicated by $\mathcal{N}(\mu, \sigma^2)$. Thus, the expression $\tilde{\varepsilon} \sim \mathcal{N}(0,1)$ denotes that $\tilde{\varepsilon}$ follows a standard normal distribution.

[56]For details to the Wiener process see Hull (2006), p. 328 ff.

[57]See Appendix 2.8.2.

using the characteristic of a Wiener process $W_0 = 0$. Using this distribution of the assets at time T from (2.56) and the definition of the Wiener process, the probability of default (2.53) can be calculated:[58]

$$
\begin{aligned}
PD &= \mathbb{P}\left(\tilde{A}_T < B\right) \\
&= \mathbb{P}\left(\ln\left(\frac{\tilde{A}_T}{A_0}\right) < \ln\left(\frac{B}{A_0}\right)\right) \\
&= \mathbb{P}\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{W}_T < \ln\left(\frac{B}{A_0}\right)\right) \\
&= \mathbb{P}\left(\tilde{\varepsilon}\cdot\sqrt{T} < \frac{\ln\left(\frac{B}{A_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma}\right) \\
&= \mathbb{P}\left(\tilde{\varepsilon} < -\frac{\ln\left(\frac{A_0}{B}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\cdot\sqrt{T}}\right) \\
&= \Phi\left(-\frac{\ln\left(\frac{A_0}{B}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\cdot\sqrt{T}}\right) \\
&=: \Phi(-\delta).
\end{aligned}
\tag{2.57}
$$

This expression is also known from the Black–Scholes formula of option pricing.[59] The variable δ is called "*distance to default*", as a high value of δ indicates a high equity buffer before a default event can happen. As can be seen in (2.57), the distance to default is higher if the relation of asset to liability value and the drift rate are high and the volatility is low. The problem of asset value models is that the asset value process is not observable and therefore the model cannot easily be calibrated. For firms listed on the stock exchange, the equity values can be observed instead. Therefore, several approaches have been developed for a transformation of equity into asset values.[60]

There also exist several extensions of the asset value model of Merton (1974). Black and Cox (1976) have introduced a *first passage model*, which means that the firm defaults when the asset value is lower than a default barrier for the *first* time and not only at the time of maturity T. In the first passage model of Longstaff and Schwartz (1995) it is assumed that the short-term risk-free interest rate is stochastic, modeled with a Vasicek process, and the risk-free interest rate is correlated with the asset value. Zhou (2001) models the asset return with a jump-diffusion process and thus introduces an additional source of uncertainty leading to empirically more

[59]See Black and Scholes (1973) and Merton (1973).

[60]See for example Bluhm et al. (2003), p. 141 ff. In the documentation of the KMV model (see Crosbie and Bohn 1999) the classical Merton approach is described for solving this problem but according to Bluhm et al. (2003), KMV uses an undisclosed, more complicated algorithm for this task.

plausible results for short-term loans. In addition to the class of asset value models, the probability of default is often determined with *reduced-form models*. In this class, a default is not determined endogenously but it is an exogenous event, and the default time is modeled as the first jump in a jump process. One of the first reduced-form models has been developed by Jarrow and Turnbull (1995).[61] Although the extensions of Merton's asset value model as well as the intensity models usually show a better empirical performance for modeling the PD, it is not necessarily problematic for the validity of the subsequently presented Vasicek model. Even if this model is based on the Merton model, the PD can be determined exogenously with any estimation method as can be seen in the subsequent section.

2.4 The Conditional Probability of Default Within the One-Factor Model of Vasicek

In contrast to the Merton model, the Vasicek model does not focus on the probability of default of a single obligor but quantifies the probability distribution of losses in a loan portfolio. Since the asset value processes and as a consequence the default events cannot be assumed to be independent of each other, a systematic factor is introduced into the model that influences all asset values in a portfolio.[62] As the stochastic interdependence between the firms is modeled by one systematic factor, the model is also called the *Vasicek one-factor model*. The systematic factor is introduced into the model by decomposing the stochastic component of the asset value process from (2.54) or (2.56) into two components that realize at a future point in time T: a systematic part $\tilde{x}$ that influences all firms within the portfolio and a firm-specific (idiosyncratic) part $\tilde{\varepsilon}_i$. Thus, the stochastic component $\tilde{W}_{i,T}$ of each obligor i in $t = T$ can be represented as

$$\tilde{W}_{i,T} = b_i \cdot \tilde{x}_T + c_i \cdot \tilde{\varepsilon}_{i,T}, \tag{2.58}$$

in which $\tilde{x}_T \sim \mathcal{N}(0, T)$ and $\tilde{\varepsilon}_{i,T} \sim \mathcal{N}(0, T)$ are independently and identically normally distributed with mean zero and standard deviation $\sqrt{T}$ for all $i \in \{1, \ldots, n\}$. The degree of the stochastic dependence to the systematic and the idiosyncratic factors is represented by the factor loadings b_i and c_i. In the context of such factor models, the stochastic component $\tilde{W}_i$, mathematically the realization of a standard Wiener process, is usually called the "standardized log-return" of a firm, since this variable results from the logarithm of the asset returns $\ln(\tilde{A}_T/A_0)$ after standardization, see (2.57). For the sake of clarity, the standardized log-returns of the assets

[61] A review of the literature regarding structural and reduced-form models can be found in Duffie and Singleton (2003) and Grundke (2003), p. 15 ff.

[62] Cf. Vasicek (1987).

will be denoted by $\tilde{a}_i$ instead of $\tilde{W}_i$ in the following. Using this notation and choosing a time period of $T = 1$ (e.g. 1 year), (2.58) can be written as

$$\tilde{a}_i = b_i \cdot \tilde{x} + c_i \cdot \tilde{\varepsilon}_i \tag{2.59}$$

with $\tilde{x} \sim \mathcal{N}(0,1)$ and $\tilde{\varepsilon}_i \sim \mathcal{N}(0,1)$. The factor loadings can be written as $b_i = \sqrt{\rho_i}$ and $c_i = \sqrt{1-\rho_i}$, where ρ_i is some constant, as this assures an expectation value of zero and a standard deviation of one of the standardized log-returns $\tilde{a}_i$:

$$\mathbb{E}(\tilde{a}_i) = \mathbb{E}\left(\sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i\right) = \sqrt{\rho_i} \cdot \mathbb{E}(\tilde{x}) + \sqrt{1-\rho_i} \cdot \mathbb{E}(\tilde{\varepsilon}_i) = 0, \tag{2.60}$$

$$\begin{aligned} \mathbb{V}(\tilde{a}_i) &= \mathbb{V}\left(\sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i\right) = \rho_i \cdot \mathbb{V}(\tilde{x}) + (1-\rho_i) \cdot \mathbb{V}(\tilde{\varepsilon}_i) \\ &= \rho_i + (1-\rho_i) = 1. \end{aligned} \tag{2.61}$$

In this model, the correlation structure of each firm i is represented by the firm-specific correlation $\sqrt{\rho_i}$ to the common factor.[63] The correlation between the logarithmic asset returns of two firms i, j, which is also called the *asset correlation*, can be expressed as $\sqrt{\rho_i \cdot \rho_j}$ or simply as ρ for the case of a homogeneous correlation structure:

$$\begin{aligned} \rho &= \operatorname{Corr}\left(\ln\left(\frac{\tilde{A}_{i,T}}{A_{i,0}}\right), \ln\left(\frac{\tilde{A}_{j,T}}{A_{j,0}}\right)\right) = \operatorname{Corr}\left(\tilde{a}_i, \tilde{a}_j\right) \\ &= \frac{\operatorname{Cov}\left(\tilde{a}_i, \tilde{a}_j\right)}{\sqrt{\mathbb{V}(\tilde{a}_i) \cdot \mathbb{V}\left(\tilde{a}_j\right)}} = \operatorname{Cov}\left(\tilde{a}_i, \tilde{a}_j\right) \\ &= \operatorname{Cov}\left(\sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i, \sqrt{\rho_j} \cdot \tilde{x} + \sqrt{1-\rho_j} \cdot \tilde{\varepsilon}_j\right) \\ &= \operatorname{Cov}\left(\sqrt{\rho_i} \cdot \tilde{x}, \sqrt{\rho_j} \cdot \tilde{x}\right) = \sqrt{\rho_i} \cdot \sqrt{\rho_j} \cdot \mathbb{V}(\tilde{x}) \\ &= \sqrt{\rho_i \cdot \rho_j}. \end{aligned} \tag{2.62}$$

As already mentioned, within the Vasicek model the probability of default does not have to be computed by the Merton model above but can be used as an exogenously given parameter PD_i.[64] Corresponding to (2.57), an obligor i defaults at $t = T$ when the latent variable $\tilde{a}_i$ falls below a default threshold d_i, which can be characterized by

$$\tilde{a}_i < d_i \quad \Leftrightarrow \quad \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i < d_i. \tag{2.63}$$

[63]The factors used in the model are not observable. Therefore, they are also called latent variables.

[64]The probability of default could either be determined by the institution itself or by a rating agency.

Against this background, the threshold d_i can be determined by the exogenous specification of PD_i:[65,66]

$$PD_i = \mathbb{P}\left(1_{\{\tilde{D}_i\}} = 1\right) = \mathbb{P}(\tilde{a}_i < d_i) = \Phi(d_i) \Leftrightarrow d_i = \Phi^{-1}(PD_i). \tag{2.64}$$

Thus, a default event $\tilde{D}_i$ of the firm i can be described by

$$\tilde{D}_i: \quad \tilde{a}_i = \sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i < \Phi^{-1}(PD_i). \tag{2.65}$$

If the loss distribution of a credit portfolio shall be computed by a Monte Carlo simulation, (2.65) can directly be implemented. In each simulation run the systematic factor as well as the idiosyncratic factors of each obligor are randomly generated. Herewith, the asset return is calculated according to (2.65). If the realization of $\tilde{a}_i$ is less than the threshold given by $\Phi^{-1}(PD_i)$, obligor i defaults. Assuming deterministic LGDs and exposures, the portfolio loss can be determined with formula (2.8) by summing up the exposure weights w_i multiplied by the loss given default LGD_i of each defaulted credit. After repeating this procedure a several thousand times and sorting the losses of the simulation runs, we obtain the portfolio loss distribution. At this point it can be seen that the model of Vasicek does not imply that the PDs are determined on the basis of Merton's asset value model of the previous section. Instead, every estimation method can be used for this purpose and only the dependence structure is specified by the model of Vasicek.

If the loss distribution or some characteristics of the distribution like the VaR or the ES shall be determined analytically, it is helpful to make use of the conditionally independence property of the asset returns. This means that for a given realization of the systematic factor, the asset returns are stochastically independent. Conditional on a realization of the systematic factor $\tilde{x} = x$, the probability of default of each obligor is

$$\begin{aligned}
\mathbb{P}\left(1_{\{\tilde{D}_i\}} = 1 | \tilde{x} = x\right) &= \mathbb{P}(\tilde{a}_i < d_i | \tilde{x} = x) \\
&= \mathbb{P}\left(\sqrt{\rho_i} \cdot \tilde{x} + \sqrt{1-\rho_i} \cdot \tilde{\varepsilon}_i < \Phi^{-1}(PD_i) | \tilde{x} = x\right) \\
&= \mathbb{P}\left(\tilde{\varepsilon}_i < \frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i} \cdot x}{\sqrt{1-\rho_i}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i} \cdot x}{\sqrt{1-\rho_i}}\right) =: p_i(x).
\end{aligned} \tag{2.66}$$

[65]The function $\Phi^{-1}(\cdot)$ stands for the inverse standard normal CDF.

[66]If the probability of default is determined by the asset value model, the default threshold d_i equals the negative distance to default $-\delta$, see (2.57).

This *conditional probability of default* $p_i(x)$ is the PD that would be assigned if the realization of the systematic factor at the horizon was known. By contrast, the unconditional probability of default reflects all information that is currently available, which means that the systematic factor is a random variable and therefore unknown. The unconditional PD equals the average value of the conditional PD across all possible realizations of the systematic factor.[67] This can be shown using the law of iterated expectations:[68]

$$\begin{aligned}\mathbb{E}(p_i(\tilde{x})) &= \mathbb{E}\left(\mathbb{P}\left(1_{\{\tilde{D}_i\}} = 1|\tilde{x}\right)\right) = \mathbb{E}\left(\mathbb{E}\left(1_{\{\tilde{D}_i\}}|\tilde{x}\right)\right)\\ &= \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \mathbb{P}\left(1_{\{\tilde{D}_i\}} = 1\right) = PD_i.\end{aligned} \tag{2.67}$$

Formula (2.66) for the conditional probability of default is sometimes called the *Vasicek formula* and is also used within the Basel framework. Details will be described in Sect. 2.7.

2.5 Measuring Credit Risk in Homogeneous Portfolios with the Vasicek Model

In order to achieve an analytical solution of the loss distribution, it is helpful to assume that the credit portfolio is homogeneous. In a homogeneous portfolio, all credits have the same PD, an identical (deterministic) LGD, the same EAD, and an identical asset correlation:[69]

$$PD_i = PD,\ LGD_i = LGD,\ EAD_i = EAD,\ \text{and}\ \rho_i = \rho \quad \forall i = 1, \ldots, n. \tag{2.68}$$

In (sub-)portfolios where the credits have similar exposures and similar risk characteristics the assumption of homogeneity should not be critical and lead to a good approximation of the loss distribution. Candidates for application of such a simplification are retail portfolios and in some cases portfolios of smaller banks.[70] In a homogeneous portfolio, a default of k credits leads to a relative loss of

$$l = \frac{k \cdot EAD \cdot LGD}{n \cdot EAD} = \frac{k}{n} \cdot LGD. \tag{2.69}$$

[67] Cf. Gordy (2003), p. 203.

[68] Cf. Franke et al. (2004), p. 41.

[69] This section is based on Vasicek (1987).

[70] Cf. Bluhm et al. (2003), p. 60.

As the defaults are exchangeable, this loss results for any k defaults. The probability of this event is

$$\mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} = k\right) = \underbrace{\binom{n}{k}}_{*} \cdot \mathbb{P}\left(\underbrace{\tilde{A}_{1,T} < B_1, \ldots, \tilde{A}_{k,T} < B_k}_{**}, \underbrace{\tilde{A}_{k+1,T} \geq B_{k+1}, \ldots, \tilde{A}_{n,T} \geq B_n}_{***}\right). \tag{2.70}$$

The expression $\tilde{A}_{i,T} < B_i$ indicates a default of firm i.[71] Therefore, the term (**) refers to a default of the first k credits, whereas the other n–k credits (***) do not default. The binomial coefficient (*) represents the number of possible combinations of k defaults out of n credits. Using the conditional independence property of Sect. 2.4, the probability of having k defaults can easily be computed within the one-factor model:[72]

$$\begin{aligned}
P_k &= \mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} = k\right) \\
&= \binom{n}{k} \cdot \mathbb{P}\left(\tilde{A}_{1,T} < B_1, \ldots, \tilde{A}_{k,T} < B_k, \tilde{A}_{k+1,T} \geq B_{k+1}, \ldots, \tilde{A}_{n,T} \geq B_n\right) \\
&= \binom{n}{k} \cdot \int\limits_{x=-\infty}^{\infty} \mathbb{P}\left(\tilde{A}_{1,T} < B_1, \ldots, \tilde{A}_{k,T} < B_k, \tilde{A}_{k+1,T} \geq B_{k+1}, \ldots, \tilde{A}_{n,T} \geq B_n \middle| \tilde{x} = x\right) d\Phi(x) \\
&= \binom{n}{k} \cdot \int\limits_{x=-\infty}^{\infty} \mathbb{P}\left(\tilde{\varepsilon}_1 < \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot x}{\sqrt{1-\rho}}, \ldots, \tilde{\varepsilon}_k < \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot x}{\sqrt{1-\rho}}, \right. \\
&\qquad \left. \tilde{\varepsilon}_{k+1} \geq \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot x}{\sqrt{1-\rho}}, \ldots, \tilde{\varepsilon}_n \geq \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot x}{\sqrt{1-\rho}}\right) d\Phi(x) \\
&= \int\limits_{x=-\infty}^{\infty} \binom{n}{k} \cdot (p(x))^k \cdot (1 - p(x))^{n-k} d\Phi(x).
\end{aligned} \tag{2.71}$$

This is also known as the *Vasicek binomial model* since the number of defaults (and the gross loss rate) of the portfolio is binomially distributed with probability $p(x)$ for a realization of the systematic factor $\tilde{x} = x$:[73]

[71] Cf. Sect. 2.3

[72] The second step is performed by using the Bayes' theorem for continuous distributions, cf. Appendix 2.8.3, and the standard normal distribution of the systematic factor.

[73] The notation $\mathcal{B}(n, p)$ indicates a binomial distribution with parameters n and p.

$$\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}|x\right) \sim \mathcal{B}(n, p(x)). \tag{2.72}$$

Hence, the conditional probability of k defaults equals

$$\mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} = k|\tilde{x} = x\right) = \binom{n}{k} \cdot (p(x))^k \cdot (1 - p(x))^{n-k}, \tag{2.73}$$

which is the integrand of (2.71).[74]

Due to the homogeneity of exposures, the corresponding loss distribution function is given as[75]

$$\begin{aligned} F^{(n)}(l) = \mathbb{P}\left(\tilde{L} \leq l\right) &= \mathbb{P}\left(\frac{1}{n} \cdot LGD \cdot \sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} \leq l\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} \leq \frac{l \cdot n}{LGD}\right) = \sum_{k=0}^{\lfloor l \cdot n/LGD \rfloor} P_k. \end{aligned} \tag{2.74}$$

With (2.71) and (2.74), the distribution can be computed via numerical integration; thus, in the case of homogeneous portfolios, there is no need for a Monte Carlo simulation. Furthermore, applying definition (2.15) and (2.19), the risk measures VaR and ES within the Vasicek binomial model can be computed, which will be named $VaR^{(n)}(l)$ and $ES^{(n)}(l)$, respectively, leading to

$$VaR_\alpha^{(n)}\left(\tilde{L}\right) = \inf\left\{ l \in \mathbb{R} | \mathbb{P}\left[\tilde{L} \leq l\right] = \sum_{k=0}^{\lfloor l \cdot n/LGD \rfloor} P_k \geq \alpha \right\}, \tag{2.75}$$

$$ES_\alpha^{(n)}\left(\tilde{L}\right) = \frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L} \cdot 1_{\left\{\tilde{L} \geq VaR_\alpha^{(n)}\right\}}\right] - VaR_\alpha^{(n)}\left[\mathbb{P}\left[\tilde{L} \geq VaR_\alpha^{(n)}\right] - (1-\alpha)\right]\right). \tag{2.76}$$

If it is assumed that the portfolio consists of an infinite number of obligors,[76] an easy-to-handle closed form solution of the loss distribution and the probability

[74] See also Gordy and Heitfield (2000).

[75] The symbolism $\lfloor x \rfloor$ is defined as in (2.43).

[76] In this case, the homogeneous portfolio is called "infinitely fine grained". See also Sect. 2.6 for further details.

density function can be achieved. According to Vasicek (1991), the resulting *limit distribution* is[77]

$$F^{(\infty)}(l) = \lim_{n\to\infty} F^{(n)}(l) = \Phi\left(\frac{1}{\sqrt{\rho}}\cdot\left(\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{l}{LGD}\right)-\Phi^{-1}(PD)\right)\right) \tag{2.77}$$

and the corresponding probability density function equals

$$f^{(\infty)}(l)=\sqrt{\frac{1-\rho}{\rho}} \cdot\exp\left(-\frac{1}{2\rho}\cdot\left[\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{l}{LGD}\right)-\Phi^{-1}(PD)\right]^2+\frac{1}{2}\left[\Phi^{-1}\left(\frac{l}{LGD}\right)\right]^2\right). \tag{2.78}$$

Both functions are visualized in Fig. 2.2 for the parameter setting $PD = 5\%$, $\rho = 20\%$, and $LGD = 100\%$. Obviously, the probability density function is

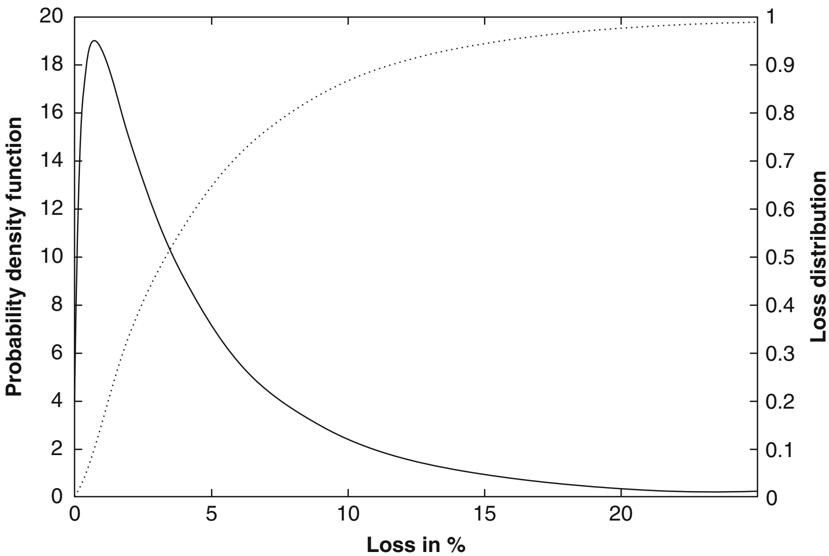

Fig. 2.2 Limiting loss distribution of Vasicek (1991)

[77] See Appendix 2.8.4.

right-skewed and the function has so-called "fat tails". Thus, the kurtosis of loss distributions is typically much higher than the kurtosis of a standard normal distribution. These characteristics reflect the relatively high probability of suffering losses that are several times higher than the expected loss.

With this resulting limit distribution, it is possible to quickly approximate the loss distribution of large subportfolios with similar risk characteristics with high accuracy. This could especially be done for subsegments of a bank's retail portfolio. Furthermore, as the distribution only depends on the PD, the LGD, and the correlation parameter, the complexity of model calibration is relatively low. Based on the loss distribution (2.77) the VaR and the ES can be computed in closed form, too:[78]

$$VaR_{\alpha}^{(\infty)}\left(\tilde{L}\right) = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\cdot\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\cdot LGD, \tag{2.79}$$

$$ES_{\alpha}^{(\infty)}\left(\tilde{L}\right) = \frac{1}{1-\alpha}\cdot\Phi_2\left(\Phi^{-1}(PD), -\Phi^{-1}(\alpha), \sqrt{\rho}\right)\cdot LGD, \tag{2.80}$$

where $\Phi_2(\cdot)$ stands for the bivariate cumulative normal distribution function. This function is defined as

$$\Phi_2\left(x, y, \varrho^2\right) := \mathbb{P}\left(\tilde{X} \le x, \tilde{Y} \le y\right) = \int\limits_{u=-\infty}^{x}\int\limits_{v=-\infty}^{y}\varphi_2(u, v)dv\,du, \tag{2.81}$$

where $\tilde{X}$, $\tilde{Y}$ are standard normal distributed random variables, which have a correlation of ϱ. The joint density function φ_2 of the bivariate standard normal distribution is defined as[79]

$$\varphi_2(u, v) := \frac{1}{2\pi\sqrt{1-\varrho^2}}\cdot\exp\left(-\frac{1}{2}\,\frac{u^2 - 2\varrho uv + v^2}{1-\varrho^2}\right). \tag{2.82}$$

2.6 Measuring Credit Risk in Heterogeneous Portfolios with the ASRF Model of Gordy

In order to achieve analytical tractability of a model that can be used for risk quantification in heterogeneous portfolios, the so-called *Asymptotic Single Risk Factor (ASRF) framework* has been developed by Gordy (2003).[80] In this framework it is assumed that

[78]See Appendix 2.8.5.

[79]Cf. Bronshtein et al. (2007), p. 779 f., especially (16.156).

[80]See also Bank and Lawrenz (2003).

(A) The portfolio is *infinitely fine-grained* and
(B) Only a *single systematic risk* factor influences the credit risk of all loans in the portfolio

Assumption (A) refers to the granularity of a portfolio that describes the impact of a single credit to the overall portfolio. In a portfolio that consists of a small number of borrowers – a *coarse-grained portfolio* – there is a relatively high impact of the firm-specific, idiosyncratic risk component. A portfolio with a high degree of name concentration is also called a "lumpy" credit portfolio. In contrast, the idiosyncratic risk vanishes in the limiting case of infinite granularity and the risk is solely a result of the uncertainty about the systematic risk factor,[81] as will be shown in the following. A portfolio is "*infinitely granular*" or "*asymptotic*" if it consists of a nearly infinite number of credits $(n \to \infty)$ with each credit having a deterministic exposure weight of negligible size. Concretely, the following conditions have to be fulfilled:[82]

$$\lim_{n\to\infty} \sum_{i=1}^{n} EAD_i = \infty, \tag{2.83}$$

$$\sum_{n=1}^{\infty} \left(\frac{EAD_n}{\sum_{j=1}^{n} EAD_j} \right)^2 < \infty. \tag{2.84}$$

Furthermore, it is assumed that all dependencies across credit events can be expressed by a set of systematic risk factors $\tilde{x}$ so that the credit events are mutually independent conditional on $\tilde{x}$.[83] This not only refers to the assumption of conditionally independent defaults but also to conditional independence of LGDs and especially of the products $(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}})$. These conditions are necessary for the applicability of the strong law of large numbers. As shown in Appendix 2.8.7, these conditions assure that the portfolio loss (almost surely) equals its conditional expectation:

$$\mathbb{P}\left(\lim_{n\to\infty} \left[\tilde{L} - \mathbb{E}\left(\tilde{L}|\tilde{x}\right) \right] = 0 \right) = 1, \tag{2.85}$$

[81]Cf. BCBS (2001a), p. 89, § 422. This effect could also be found for the limiting distribution of the Vasicek binomial model, see Sect. 2.5.

[82]Cf. Bluhm et al. (2003), p. 87 ff.

[83]Assumption (B), the existence of only a single systematic risk factor, is not needed at this stage.

which is usually much easier to calculate than the unconditional loss distribution.[84] As demonstrated in Appendix 2.8.8, (2.83) and (2.84) also assure that[85]

$$\lim_{n\to\infty} \sum_{i=1}^{n} {w_i}^2 = 0. \tag{2.86}$$

Thus, the weight of each exposure must be negligible. This formulation is directly related to the Herfindahl–Hirschmann Index (HHI), a common measure for indicating the degree of concentration:[86]

$$HHI = \sum_{i=1}^{n} {w_i}^2 = \frac{1}{n^*}. \tag{2.87}$$

In contrast to the actual number of credits n, the variable n^* is the so-called "effective number" of credits. In a homogeneous portfolio, which has the least possible exposure concentration for a given number of credits, n and n^* are identical. Hence, n^* can be interpreted as the number of credits in a homogeneous portfolio with the equivalent degree of name concentration risk. (2.86) can therewith be formulated as

$$\lim_{n\to\infty} \sum_{i=1}^{n} {w_i}^2 = \lim_{n\to\infty} \frac{1}{n^*} = 0, \tag{2.88}$$

which shows that it is not enough that the *actual* number of credits goes to infinity but the *effective* number of credits must go to infinity.

Using property (2.85) the VaR can be written as[87]

$$\lim_{n\to\infty} VaR_\alpha\left(\tilde{L}\right) = VaR_\alpha\left(\mathbb{E}\left[\tilde{L}|\tilde{x}\right]\right). \tag{2.89}$$

Additionally, Gordy (2003) has introduced assumption (B), which states that there is only a single risk factor that influences the credit risk of all loans. Thus, it is assumed that there exist no sector-specific risk factors such as industry-specific or

---

[84] For ease of notation, the convergence of a sequence X_n towards X with probability one is indicated by $\lim_{n\to\infty} X_n = X$ instead of $\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$ in the following.

[85] This is the result of Kronecker's Lemma, see Appendix 2.8.8, which is also needed to proof the strong law of large numbers presented in Appendix 2.8.7. This condition has also been formulated by Vasicek (2002), p. 160.

[86] See BCBS (2001a), p. 97, § 459 and Gordy (2003). The HHI was used in this earlier version of the Basel framework for mapping a heterogeneous portfolio into a comparable homogeneous portfolio.

[87] See Gordy (2003), p. 206 ff.

geographical risk factors and consequently no concentrations in specific sectors. If assumptions (A) and (B) are fulfilled, the following identity holds:[88]

$$VaR_\alpha\left(\mathbb{E}\left[\tilde{L}|\tilde{x}\right]\right) = \mathbb{E}\left(\tilde{L}|\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \tag{2.90}$$

This leads to the important proposition

$$VaR_\alpha^{(\mathrm{ASRF})} = \lim_{n\to\infty} VaR_\alpha\left(\tilde{L}\right) = VaR_\alpha\left(\mathbb{E}\left[\tilde{L}|\tilde{x}\right]\right) = \mathbb{E}\left(\tilde{L}|\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \tag{2.91}$$

As a result of the conditional independence of all credit events, this proposition can be written as

$$\begin{aligned} VaR_\alpha^{(\mathrm{ASRF})} &= \mathbb{E}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} |\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right) \\ &= \sum_{i=1}^{n} \mathbb{E}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} |\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \end{aligned} \tag{2.92}$$

It is obvious that the risk contribution of a single credit is equal to its conditional expected loss and is therefore constant, regardless of the concrete portfolio to which the credit is added. This characteristic is also called *portfolio-invariance*. This can be explained by the fact that each individual claim does not cause any (further) diversification effect, since the portfolio has already reached the highest possible degree of diversification. A further important implication is that the VaR of a portfolio is exactly additive because the expected value is exactly additive as well. Consequently, the axiom of subadditivity holds and the VaR is a coherent risk measure under the assumptions described above.[89]

The corresponding expression for the risk measure ES is[90]

$$\lim_{n\to\infty} ES_\alpha\left(\tilde{L}\right) = ES_\alpha\left(\mathbb{E}\left(\tilde{L}|\tilde{x}\right)\right) \tag{2.93}$$

leading to

$$ES_a^{(\mathrm{ASRF})}\left(\tilde{L}\right) = ES_\alpha\left(\sum_{i=1}^{n} \mathbb{E}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} |\tilde{x}\right)\right). \tag{2.94}$$

[88]See Appendix 2.8.9. The slightly different result concerning the confidence level results from a different definition of the systematic factor. Gordy (2003) assumes that the expected loss is monotonously *increasing* in x, whereas here it is assumed that the expected loss is monotonously *decreasing* in x. In other words, large values of x indicate a good economic condition in this setting.

[89]Cf. Sect. 2.2.2.

[90]See Appendix 2.8.10.

Although the equivalent to (2.91) cannot be formulated for the ES in general form, many specified single-factor models still allow to determine the ES analytically.[91]

2.7 Measuring Credit Risk Within the IRB Approach of Basel II

The IRB Approach of Basel II is based on both the ASRF framework of Gordy (2003) and the conditional probability of default resulting from Vasicek (1987). Under the assumptions of the ASRF framework, it has been shown that the VaR is given as

$$VaR_{\alpha}^{(\mathrm{ASRF})}\left(\tilde{L}\right) = \sum_{i=1}^{n} \mathbb{E}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\left\{\tilde{D}_i\right\}} | \tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \tag{2.95}$$

The confidence level is chosen as $\alpha = 0.999$ in the Basel framework.[92] Furthermore, the conditional probability of default is specified to

$$\mathbb{P}\left(1_{\left\{\tilde{D}_i\right\}} = 1 | \tilde{x} = x\right) = \mathbb{E}\left(1_{\left\{\tilde{D}_i\right\}} | \tilde{x} = x\right) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i} \cdot x}{\sqrt{1-\rho_i}}\right), \tag{2.96}$$

which is a result of the Vasicek one-factor model. Recalling the standard normal distribution of the systematic factor, the VaR can be written as

$$\begin{aligned} VaR_{0.999}^{(\mathrm{Basel})}\left(\tilde{L}\right) &= \sum_{i=1}^{n} \mathbb{E}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\left\{\tilde{D}_i\right\}} | \tilde{x} = VaR_{0.001}(\tilde{x})\right) \\ &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left(\widetilde{LGD}_i | \tilde{x} = \Phi^{-1}(0.001)\right) \cdot \mathbb{E}\left(1_{\left\{\tilde{D}_i\right\}} | \tilde{x} = \Phi^{-1}(0.001)\right) \\ &= \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left(\widetilde{LGD}_i | \tilde{x} = -\Phi^{-1}(0.999)\right) \cdot \Phi\left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i} \cdot \Phi^{-1}(0.999)}{\sqrt{1-\rho_i}}\right). \end{aligned} \tag{2.97}$$

This is the core element of the Basel II framework, even if there are some minor differences between the formula above and the concrete capital requirements. These differences are:

[91] Cf. Gordy (2003), p. 219.

[92] From the second to the third consultative document of the Basel framework, the confidence level was changed from $\alpha = 0.995$ to $\alpha = 0.999$; cf. BCBS (2001a, 2003a).

- The capital requirements are only applied to the *Unexpected Loss* (UL), which is the difference of VaR and EL. This is due to the fact that the expected loss is already accounted for in the provisions. As the loan loss provisioning reduces the equity, a capital requirement which includes the expected loss would require this capital amount twice.[93]
- The LGD-specific term of (2.97) shows that the expected LGD under the specified conditions of a VaR scenario is needed. The regulatory formula simply uses the notation "LGD" in the VaR term as well as in the expected loss term. However, this does not mean that the expected LGD has to be inserted. If an institution uses own LGD estimates, these have to "reflect economic downturn conditions where necessary to capture the relevant risks".[94] This LGD is also called "*Downturn LGD*" (DLGD). A background note on LGD quantification clarifies that the downturn LGD is at least in principle meant in terms of the conditional LGD of (2.97). But as a concrete quantification and validation of downturn LGDs in the sense above is found to be "not operationally feasible given the current state of practice in this area", there is no regulatory function that transforms the unconditional into a conditional LGD and also no explicit demand for LGD quantification in a 99.9% scenario.[95]
- The PD in the formula above refers to the 1-year probability of default. In practice, many loans have an effective maturity M_i that can substantially differ from 1 year, especially towards longer maturities. As a long-term loan is usually considered as more risky than a short-term loan, this shall also be reflected in the capital requirement. Therefore a so-called *Maturity Adjustment* is implemented as a factor in the Basel II capital rules.[96]
- The overall level of minimum capital requirements of the model above is calibrated to a regulatory desired magnitude by introducing a *Scaling Factor* (SF), which has to be multiplied to the result of the model itself. This factor is set

[93]Because of this argument, the former version of the capital rules, which had the VaR and not the UL as capital requirement, were changed; cf. BCBS (2001a). The problem is that the regulatory rules and the different accounting standards are not fully consistent. Therefore, a bank has to compare the amount of total eligible provisions with the total expected losses amount. If the EL exceeds the provisions, the difference has to be deducted such that it is guaranteed that the total capital amount captures both the UL and the EL; cf. BCBS (2005a), § 43.

[94]BCBS (2005a), § 468.

[95]Cf. BCBS (2004a). Interestingly, the supervisors in the United States proposed a concrete function for mapping the ELGD into the DLGD: $DLGD = 0.08 + 0.92 \cdot ELGD$. Thus, the downturn LGD was a linear mapping from [0%, 100%] to [8%, 100%]. However, in the final rule this supervisory mapping function is not included because of several points of criticism. Nevertheless, the agencies still believe that the formula is an appropriate way to deal with problems in estimating downturn LGDs; cf. FDIC (2007), Sect. III.B.3, p. 69310. However, there is no direct link between this mapping function and the conditional LGD as presented in (2.97).

[96]Cf. Heithecker (2007), p. 31 f., p. 57 ff., and p. 235 ff., for details regarding the maturity adjustment including an outline of the corresponding literature.

to $SF = 1.06$, which is based on the data of the Quantitative Impact Study 3 (QIS 3).[97]

Taking all these points together, the *capital requirement for each credit* under Basel II (in absolute terms) can be expressed as[98]

$$\begin{aligned} UL_{\text{abs},i}^{(\text{Basel})} &= VaR_{\text{abs},i}^{(\text{Basel})} - EL_{\text{abs},i}^{(\text{Basel})} \\ &= EAD_i \cdot \left[DLGD_i \cdot \Phi\left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i} \cdot \Phi^{-1}(0.999)}{\sqrt{1-\rho_i}} \right) - ELGD_i \cdot PD_i \right] \\ &\quad \cdot \frac{1 + (M_i - 2.5) \cdot b}{1 - 1.5 \cdot b} \cdot 1.06 \end{aligned} \tag{2.98}$$

with $b = [0.11852 - 0.05478 \cdot \ln(PD_i)]^2$. Furthermore, the *correlation parameter* is specified by the regulatory framework. Dependent on the asset class (and for some asset classes dependent on the PD and revenue, too), the correlation parameter is between 3% and 24%.[99] For corporate, sovereign, and bank exposures (C,S,B), ρ_i is between 12% (if the PD is very high) and 24% (if the PD is very low):[100]

$$\rho_i^{(\text{C,S,B})} = 0.12 \cdot \frac{1 - \exp(-50 \cdot PD_i)}{1 - \exp(-50)} + 0.24 \cdot \left(1 - \frac{1 - \exp(-50 \cdot PD_i)}{1 - \exp(-50)} \right). \tag{2.99}$$

For small- and medium-sized entities (SMEs), a firm-size adjustment is made. Depending on the total annual sales S_i (in millions of Euros), the correlation parameter will be reduced linearly between 4% (for $S_i \leq 5$) and 0% (for $S_i = 50$):[101]

$$\rho_i^{(\text{SME})} = \rho_i^{(\text{C,S,B})} - 0.04 \cdot \left(1 - \frac{\max(S, 5) - 5}{45} \right), \tag{2.100}$$

---

[97] In total, 365 banks participated in the study, which focused on the impact of the Basel II proposals on the minimum capital requirements compared to Basel I; cf. BCBS (2003b).

[98] Cf. BCBS (2005a), § 272, § 273, § 328, § 329, and § 330. The maturity adjustment is only applied to corporate, sovereign, and bank exposures, including small- and medium-sized entities (SMEs). This can also be interpreted as a fixed maturity of $M_i = 1$ year for retail exposures.

[99] For internal purposes a bank could measure ρ from default series or from equity values; cf. Gordy and Heitfield (2002), Düllmann and Trapp (2005), or Lopez (2004). The results for estimating ρ from portfolio data may differ from the correlations given in Basel II, see e.g. Düllmann and Scheule (2003) or Dietsch and Petey (2002), but overall the parameters given in Basel II are reasonable, see especially Lopez (2004).

[100] Cf. BCBS (2005a), § 272. The concrete definition of corporate exposures can be found in BCBS (2005a), § 218 ff.; sovereign and bank exposures are defined in § 229 and § 230.

[101] Cf. BCBS (2005a), § 273.

which leads to a reduction of capital requirements for SMEs. For residential mortgage exposures the correlation is fixed to 15%,[102] for qualifying revolving retail exposures to 4%,[103] and for other retail exposures the correlation parameter is between 3% and 16%:[104]

$$\rho_i^{(\text{Retail})} = 0.03 \cdot \frac{1-\exp(-35 \cdot PD_i)}{1-\exp(-35)} + 0.16 \cdot \left(1 - \frac{1-\exp(-35 \cdot PD_i)}{1-\exp(-35)}\right). \tag{2.101}$$

Taking (2.98) into consideration, the parameters *EAD*, *PD*, *LGD*, and *M* have to be determined. As the complexity of these estimations and the data requirement would be too high for many banks, there exist two versions of the IRB Approach for corporate, sovereign, and bank exposures, as mentioned in Sect. 2.1. In the *Advanced IRB Approach*, all of these parameters have to be estimated by the bank. In the *Foundation IRB Approach*, the LGD and maturity are given by the regulatory rules. Furthermore, only the current outstandings and the commitments have to be determined by the bank, the credit conversion factor and therefore the EAD does not have to be estimated. Thus, under the Foundation Approach, the only parameter that has to be estimated by the bank is the PD.[105] However, for retail exposures, there is no distinction between a Foundation and Advanced IRB Approach. In the *IRB-Retail-Approach*, the parameters EAD, PD, and LGD have to be estimated by the bank.[106] However, in contrast to the IRB Approaches of the other asset classes, in the IRB-Retail-Approach it is allowed to pool credits with similar characteristics such as risk characteristics, collaterals and exposures.[107] As the parameter estimates for the retail portfolio can be based on these risk pools instead of individual borrower grades,[108] the minimum complexity of the IRB-Retail-Approach is significantly lower than of the Advanced IRB Approach.

[102] Cf. BCBS (2005a), § 328.

[103] Cf. BCBS (2005a), § 329.

[104] Cf. BCBS (2005a), § 330.

[105] Cf. BCBS (2005a), § 246 f.

[106] Cf. BCBS (2005a), § 252. The definition of retail exposures can be found in BCBS (2005a), § 231 ff.

[107] Cf. BCBS (2005a), § 401 f.

[108] Cf. BCBS (2005a), § 446.

2.8 Appendix

2.8.1 Alternative Representation of the ES as an Indicator Function

Proposition. *The definition of the ES (2.19) is equal to (2.29):*[109]

$$\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha\}}\right]-q_\alpha\left[\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)\right]\right)=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right]\right) \tag{2.102}$$

with

$$1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}=\begin{cases}1_{\{\tilde{L}\geq q_\alpha\}} & \text{if } \mathbb{P}\left[\tilde{L}=q_\alpha\right]=0,\\ 1_{\{\tilde{L}\geq q_\alpha\}}-\frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}\cdot 1_{\{\tilde{L}=q_\alpha\}} & \text{if } \mathbb{P}\left[\tilde{L}=q_\alpha\right]>0.\end{cases} \tag{2.103}$$

Proof. For the case $\mathbb{P}\left[\tilde{L}=q_\alpha\right]=0$, the left-hand side immediately equals the right-hand side of (2.102). Therefore, only the case $\mathbb{P}\left[\tilde{L}=q_\alpha\right]>0$ is analyzed:

$$\begin{aligned}ES_\alpha\left(\tilde{L}\right)&=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha\}}\right]-q_\alpha\left[\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)\right]\right)\\&=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha\}}\right]-\tilde{L}\cdot 1_{\{\tilde{L}=q_\alpha\}}\cdot\left(\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)\right)\right)\\&=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha\}}-\frac{\tilde{L}\cdot 1_{\{\tilde{L}=q_\alpha\}}\cdot\left(\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)\right)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}\right]\right)\\&=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot\left(1_{\{\tilde{L}\geq q_\alpha\}}-\frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}1_{\{\tilde{L}=q_\alpha\}}\right)\right]\right)\\&=\frac{1}{1-\alpha}\left(\mathbb{E}\left[\tilde{L}\cdot 1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right]\right),\end{aligned} \tag{2.104}$$

which is the proposed right-hand side of (2.102).

Additionally, we want to show some properties of the function $1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}$, which are useful for analyzing the axioms of coherency. The expected value of this variable is

[109] Cf. Acerbi et al. (2001), p. 8 f.

$$
\begin{aligned}
\mathbb{E}\left[1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\right] &= \mathbb{E}\left[1_{\{\tilde{L}\geq q_\alpha\}} - \frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}\cdot 1_{\{\tilde{L}=q_\alpha\}}\right] \\
&= \mathbb{E}\left[1_{\{\tilde{L}\geq q_\alpha\}}\right] - \mathbb{E}\left[\frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]}\cdot 1_{\{\tilde{L}=q_\alpha\}}\right] \\
&= \mathbb{P}\left[\tilde{L}\geq q_\alpha\right] - \left(\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)\right) \\
&= 1-\alpha.
\end{aligned}
\tag{2.105}
$$

Moreover, we want to show that $1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\in[0,\ 1]$. For $\tilde{L}\neq q_\alpha(\tilde{L})$ this is obvious by the definition of the indicator function. However, for $\tilde{L}=q_\alpha(\tilde{L})$, the variable is given as

$$
\begin{aligned}
1^{\alpha}_{\{\tilde{L}\geq q_\alpha\}}\Big|_{\tilde{L}=q_\alpha} &= 1-\frac{\mathbb{P}\left[\tilde{L}\geq q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]} \\
&= 1-\frac{\mathbb{P}\left[\tilde{L}>q_\alpha\right]+\mathbb{P}\left[\tilde{L}=q_\alpha\right]-(1-\alpha)}{\mathbb{P}\left[\tilde{L}=q_\alpha\right]} \\
&= \frac{-\mathbb{P}\left[\tilde{L}>q_\alpha\right]+(1-\alpha)}{\mathbb{P}\left[\tilde{L}\leq q_\alpha\right]-\mathbb{P}\left[\tilde{L}<q_\alpha\right]} \\
&= \frac{\mathbb{P}\left[\tilde{L}\leq q_\alpha\right]-\alpha}{\mathbb{P}\left[\tilde{L}\leq q_\alpha\right]-\mathbb{P}\left[\tilde{L}<q_\alpha\right]}\in[0,\ 1],
\end{aligned}
\tag{2.106}
$$

because of $\mathbb{P}\left[\tilde{L}<q_\alpha\right]\leq\alpha\leq\mathbb{P}\left[\tilde{L}\leq q_\alpha\right]$.

2.8.2 Application of Itô's Lemma

An Itô-process is given as

$$dA_t = a(A_t,t)dt + b(A_t,t)dW_t. \tag{2.107}$$

With $a(A_t,t)=\mu\cdot A_t$ and $b(A_t,t)=\sigma\cdot A_t$, we get the stochastic process of the asset value (see (2.54))

$$dA_t = \mu\cdot A_t\,dt + \sigma\cdot A_t\,dW_t. \tag{2.108}$$

Therefore, the asset value follows an Itô-process and Itô's Lemma can be applied in order to determine $dY_t = d\ln A_t$. When Y_t is a function of A_t and t, so we write $Y_t = g(A_t,t)$, Itô's Lemma shows that[110]

[110] Cf. Hull (2006), p. 273 f.

$$
\begin{aligned}
dY_t &= dg(A_t, t) \\
&= \left(\frac{dg}{dA_t} \cdot a(A_t, t) + \frac{dg}{dt} + \frac{1}{2}\frac{d^2 g}{dA_t{}^2} \cdot b^2(A_t, t)\right) dt + \frac{dg}{dA_t} \cdot b(A_t, t)\, dW_t.
\end{aligned} \tag{2.109}
$$

With $dY_t = d\ln A_t$, $a(A_t, t) = \mu \cdot A_t$, and $b(A_t, t) = \sigma \cdot A_t$, this leads to

$$
\begin{aligned}
dY_t &= d\ln A_t \\
&= \left(\frac{1}{A_t} \cdot \mu \cdot A_t + 0 + \frac{1}{2}\left(-\frac{1}{A_t{}^2}\right)(\sigma \cdot A_t)^2\right) dt + \frac{1}{A_t}(\sigma \cdot A_t) dW_t \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t.
\end{aligned} \tag{2.110}
$$

2.8.3 Application of Bayes' Theorem for Continuous Distributions

The definition of probability density functions and Bayes' theorem for continuous distributions lead to[111]

$$
\begin{aligned}
\mathbb{P}(\tilde{y} < u) &= \int_{y=-\infty}^{u} f_Y(y)\, dy \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{u} f_{X,Y}(x, y)\, dy\, dx \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{u} f_Y(y|x)\, dy\, f_X(x)\, dx \\
&= \int_{x=-\infty}^{\infty} \mathbb{P}(\tilde{y} < u|x) f_X(x)\, dx.
\end{aligned} \tag{2.111}
$$

Thus, using $\frac{dF_X(x)}{dx} = f_X(x)$ we get

$$
\mathbb{P}(\tilde{y} < u) = \int_{x=-\infty}^{\infty} \mathbb{P}(\tilde{y} < u|x)\, dF_X(x). \tag{2.112}
$$

[111] Cf. Tarantola (2005), p. 20.

2.8.4 Limit Distribution and Probability Density Function in the Vasicek Model

In the following, the integral of the distribution (2.74) of the binomial model shall be solved for the limit $n \to \infty$:[112]

$$\begin{aligned} F^{(\infty)}(l) &= \lim_{n\to\infty} F^{(n)}(l) \\ &= \lim_{n\to\infty} \sum_{k=0}^{\lfloor l\cdot n/LGD\rfloor} \int_{x=-\infty}^{\infty} \binom{n}{k} \cdot (p(x))^k \cdot (1-p(x))^{n-k} d\Phi(x) \end{aligned} \tag{2.113}$$

with

$$p(x) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\cdot x}{\sqrt{1-\rho}}\right). \tag{2.114}$$

Using $p(x) =: s$ and the identity $\Phi(-y) = 1 - \Phi(y)$, it follows

$$\begin{aligned} s &= \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}\cdot x}{\sqrt{1-\rho}}\right) \\ &\Leftrightarrow \sqrt{1-\rho}\cdot\Phi^{-1}(s) = \Phi^{-1}(PD) - \sqrt{\rho}\cdot x \\ &\Leftrightarrow x = -\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\cdot\Phi^{-1}(s) - \Phi^{-1}(PD)\right) \\ &\Leftrightarrow \Phi(x) = 1 - \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\cdot\Phi^{-1}(s) - \Phi^{-1}(PD)\right)\right) =: 1 - W(s). \end{aligned} \tag{2.115}$$

Using $d\Phi(x) = d(1-W(s)) = -dW(s)$ and $\lim_{x\to-\infty} s = \lim_{x\to-\infty} p(x) = 1$ as well as $\lim_{x\to\infty} s = \lim_{x\to\infty} p(x) = 0$, the integral (2.113) can be written as

$$\begin{aligned} F^{(\infty)}(l) &= \lim_{n\to\infty} \int_{s=1}^{0} \sum_{k=0}^{\lfloor l\cdot n/LGD\rfloor} \binom{n}{k} \cdot s^k \cdot (1-s)^{n-k} \cdot (-1) dW(s) \\ &= \int_{s=0}^{1} \lim_{n\to\infty} \sum_{k=0}^{\lfloor l\cdot n/LGD\rfloor} \binom{n}{k} \cdot s^k \cdot (1-s)^{n-k} dW(s). \end{aligned} \tag{2.116}$$

[112] The derivation is based on Vasicek (1991). In contrast to the original paper the derivation is not restrained to the gross loss but includes deterministic $LGD \neq 1$.

The integrand of (2.116) is binomially distributed. According to the central limit theorem of Lindberg-Lévy or the special case for binomial distributions of Moivre–Laplace, this distribution converges to a normal distribution for $n \to \infty$:[113]

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{\lfloor l\cdot n/LGD \rfloor} \binom{n}{k} \cdot s^k \cdot (1-s)^{n-k} &= \lim_{n\to\infty} \Phi\left(\frac{n \cdot l/LGD - n \cdot s}{\sqrt{n \cdot s \cdot (1-s)}}\right) \\
&= \lim_{n\to\infty} \Phi\left(\frac{\sqrt{n}}{\sqrt{s \cdot (1-s)}}\left(\frac{l}{LGD} - s\right)\right) \\
&= \begin{cases} \Phi(\infty) = 1 & \text{if } l/LGD > s, \\ \Phi(0) = 1/2 & \text{if } l/LGD = s, \\ \Phi(-\infty) = 0 & \text{if } l/LGD < s. \end{cases}
\end{aligned} \tag{2.117}$$

Therefore, using $W(0) = \Phi(-\infty) = 0$,[114] the distribution (2.116) is equal to

$$\begin{aligned}
F^{(\infty)}(l) &= \int_{s=0}^{1} \lim_{n\to\infty} \sum_{k=0}^{\lfloor l\cdot n/LGD \rfloor} \binom{n}{k} \cdot s^k \cdot (1-s)^{n-k} dW(s) \\
&= \int_{s=0}^{l/LGD} 1 \, dW(s) \\
&= W(s)\big|_{s=0}^{l/LGD} \\
&= W\left(\frac{l}{LGD}\right) \\
&= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho} \cdot \Phi^{-1}\left(\frac{l}{LGD}\right) - \Phi^{-1}(PD)\right)\right).
\end{aligned} \tag{2.118}$$

The corresponding probability density function $f^{(\infty)}(l)$ is the first derivative of $F^{(\infty)}(l)$. With $d\Phi(y)/dy = \varphi(y)$, $d\Phi^{-1}(y)/dy = 1/\varphi(\Phi^{-1}(y))$, and $\varphi(y) = (1/\sqrt{2\pi}) \cdot \exp(-y^2/2)$ this leads to

[113] See Billingsley (1995), p. 357 f.

[114] Cf. (2.115).

$$\begin{aligned} f^{(\infty)}(l) &= \frac{dF^{(\infty)}(l)}{dl} \\ &= \varphi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{l}{LGD}\right)-\Phi^{-1}(PD)\right)\right)\cdot\sqrt{\frac{1-\rho}{\rho}}\cdot\frac{1}{\varphi\left(\Phi^{-1}\left(\frac{l}{LGD}\right)\right)} \\ &= \sqrt{\frac{1-\rho}{\rho}}\exp\left(-\frac{1}{2\rho}\cdot\left[\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{l}{LGD}\right)-\Phi^{-1}(PD)\right]^2\right. \\ &\quad \left.+\frac{1}{2}\left[\Phi^{-1}\left(\frac{l}{LGD}\right)\right]^2\right). \end{aligned} \tag{2.119}$$

2.8.5 VaR and ES of the Limit Distribution in the Vasicek Model

According to (2.14), the VaR for continuous distributions can be expressed as

$$VaR_\alpha(\tilde{L}) = F_L^{-1}(\alpha). \tag{2.120}$$

Thus, corresponding to distribution (2.77), the VaR can be computed as follows:

$$\begin{aligned} F^{(\infty)}\left(VaR_\alpha^{(\infty)}(\tilde{L})\right) &= \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{VaR_\alpha^{(\infty)}(\tilde{L})}{LGD}\right)-\Phi^{-1}(PD)\right)\right)\overset{!}{=}\alpha \\ \Leftrightarrow \sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{VaR_\alpha^{(\infty)}(\tilde{L})}{LGD}\right) &= \Phi^{-1}(PD)+\sqrt{\rho}\cdot\Phi^{-1}(\alpha) \\ \Leftrightarrow VaR_\alpha^{(\infty)}(\tilde{L}) &= \Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}\cdot\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\cdot LGD. \end{aligned} \tag{2.121}$$

In order to determine the ES, the representation of (2.20) is used:

$$ES_\alpha(\tilde{L}) = \frac{1}{1-\alpha}\int_{u=\alpha}^{1} q^u du. \tag{2.122}$$

With (2.121) and using the substitution $y := -\Phi^{-1}(u)$ so that $du/dy = -\varphi(y)$, $y(u=\alpha) = -\Phi^{-1}(\alpha)$ and $y(u=1) = -\Phi^{-1}(1) = -\infty$, this leads to

$$ES_\alpha^{(\infty)}(\tilde{L}) = \frac{1}{1-\alpha}\int_{u=\alpha}^{1} LGD\cdot\Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}\cdot\Phi^{-1}(u)}{\sqrt{1-\rho}}\right)du$$

$$
\begin{aligned}
&= \frac{1}{1-\alpha} \cdot LGD \cdot \int_{y=-\Phi^{-1}(\alpha)}^{-\infty} \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y}{\sqrt{1-\rho}}\right) \cdot (-1) \cdot \varphi(y) dy \\
&= \frac{1}{1-\alpha} \cdot LGD \cdot \int_{y=-\infty}^{-\Phi^{-1}(\alpha)} \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y}{\sqrt{1-\rho}}\right) \cdot \varphi(y) dy.
\end{aligned} \tag{2.123}
$$

With the identity[115]

$$
\int_{y=-\infty}^{z} \Phi\left(\frac{x - a \cdot y}{\sqrt{1-a^2}}\right) \cdot \varphi(y) dy = \Phi_2(x, z, a), \tag{2.124}
$$

where $\Phi_2(\cdot)$ is the bivariate cumulative normal distribution as defined in (2.81), (2.123) can be expressed as[116]

$$
ES_{\alpha}^{(\infty)}\left(\tilde{L}\right) = \frac{1}{1-\alpha} \cdot LGD \cdot \Phi_2\left(\Phi^{-1}(PD), -\Phi^{-1}(\alpha), \sqrt{\rho}\right). \tag{2.125}
$$

2.8.6 Alternative Representation of the Bivariate Normal Distribution

Proposition. *The bivariate normal distribution can be represented as*

$$
\int_{y=-\infty}^{z} \Phi\left(\frac{x - a \cdot y}{\sqrt{1-a^2}}\right) \cdot \varphi(y) dy = \Phi_2(x, z, a). \tag{2.126}
$$

Proof. From

$$
\int_{y=-\infty}^{z} \Phi\left(\frac{x - a \cdot y}{\sqrt{1-a^2}}\right) \cdot \varphi(y) dy = \frac{1}{2\pi} \int_{y=-\infty}^{z} \int_{u=-\infty}^{\frac{x-a \cdot y}{\sqrt{1-a^2}}} \exp\left(-\frac{1}{2} y^2\right) \cdot \exp\left(-\frac{1}{2} u^2\right) du\, dy \tag{2.127}
$$

and using the substitution $u := \frac{w - a \cdot y}{\sqrt{1-a^2}}$ so that $\frac{du}{dw} = \frac{1}{\sqrt{1-a^2}}$, $w(u = -\infty) = -\infty$ and $w\left(u = \frac{x-a \cdot y}{\sqrt{1-a^2}}\right) = x$, we obtain[117]

[115] See Appendix 2.8.6.

[116] See also Pykhtin (2004).

[117] The definition of the bivariate standard normal CDF used in the last step is given in (2.81).

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{y=-\infty}^{z}\int_{u=-\infty}^{\frac{x-a\cdot y}{\sqrt{1-a^2}}}\exp\left(-\frac{1}{2}y^2\right)\cdot\exp\left(-\frac{1}{2}u^2\right)du\,dy\\
&=\frac{1}{2\pi}\int_{y=-\infty}^{z}\int_{w=-\infty}^{x}\exp\left(-\frac{1}{2}y^2\right)\cdot\exp\left(-\frac{1}{2}\left(\frac{w-a\cdot y}{\sqrt{1-a^2}}\right)^2\right)\cdot\frac{1}{\sqrt{1-a^2}}dw\,dy\\
&=\frac{1}{2\pi\sqrt{1-a^2}}\int_{y=-\infty}^{z}\int_{w=-\infty}^{x}\exp\left(-\frac{1}{2}\left(y^2+\frac{w^2-2ayw+a^2y^2}{1-a^2}\right)\right)dw\,dy\\
&=\frac{1}{2\pi\sqrt{1-a^2}}\int_{y=-\infty}^{z}\int_{w=-\infty}^{x}\exp\left(-\frac{1}{2(1-a^2)}\left(y^2-2ayw+w^2\right)\right)dw\,dy\\
&=:\Phi_2(x,z,a).
\end{aligned}
\tag{2.128}
$$

2.8.7 Application of the Strong Law of Large Numbers

Proposition. *The portfolio loss is almost surely equal to the conditional expected loss*

$$
\mathbb{P}\left(\lim_{n\to\infty}\left[\tilde{L}-\mathbb{E}\left(\tilde{L}|\tilde{x}\right)\right]=0\right)=1 \tag{2.129}
$$

under the conditions of infinite granularity (2.83) and (2.84).[118]

Proof. The proof is based upon a version of the strong law of large numbers. For an independent random sequence $\tilde{Z}_i$ the following almost sure convergence holds[119]

$$
\mathbb{P}\left(\lim_{n\to\infty}\left[\frac{1}{a_n}\sum_{i=1}^{n}\tilde{Z}_i\right]=0\right)=1 \quad \forall x\in\mathbb{R} \tag{2.130}
$$

if

$$
\lim_{n\to\infty}a_n=\infty \tag{2.131}
$$

[118]The following proof is similar to Gordy (2003), p. 223 f. and Bluhm et al. (2003), p. 88 f.

[119]See Petrov (1996), p. 209, Theorem 6.6.

and

$$\sum_{n=1}^{\infty}\left(\frac{\mathbb{V}(\tilde{Z}_n)}{{a_n}^2}\right)<\infty. \tag{2.132}$$

The random sequence $\tilde{Z}_i$ can be defined as $\tilde{Z}_i := EAD_i \cdot \left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} - \mathbb{E}\left[\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}|\tilde{x}\right]\right)$. As it is required that the $\tilde{Z}_i$s are independent, the strong law of large numbers is applied conditional on the realization of the systematic factor $\tilde{x} = x$. Under this condition, the products $\left(\widetilde{LGD}_i \cdot 1_{\{D_i\}}\right)$ are independent by assumption and therefore the $\tilde{Z}_i$'s are independent as well. Defining $a_n := \sum_{j=1}^{n} EAD_j$, the condition (2.131) directly follows from the first granularity assumption (2.83). In order to check the second condition, the boundedness of $\tilde{Z}_n$ is analyzed. The loss variable $1_{\{\tilde{D}_n\}}$ only takes the values one and zero. The LGD is assumed to be in the interval $[-1,\ 1]$.[120] As a consequence, the product $\left(\widetilde{LGD}_n \cdot 1_{\{D_n\}}\right)$ is bounded to $[-1,\ 1]$ and $\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} - \mathbb{E}\left[\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}|\tilde{x}\right]\right)$ is restricted to $[-2,\ 2]$, leading to $\mathbb{V}(\tilde{Z}_n) \leq 4 \cdot {EAD_n}^2$. Therefore, the second condition (2.132) can be written as

$$\sum_{n=1}^{\infty}\left(\frac{\mathbb{V}(\tilde{Z}_n)}{{a_n}^2}\right) \leq \sum_{n=1}^{\infty} 4 \cdot \left(\frac{EAD_n}{\sum\limits_{j=1}^{n} EAD_j}\right)^2 < \infty. \tag{2.133}$$

The last expression is valid due to the second granularity condition (2.84). Thus, the strong law of large numbers (2.130) can be applied. With

$$\begin{aligned}
\frac{1}{a_n}\sum_{i=1}^{n}\tilde{Z}_i &= \frac{1}{\sum\limits_{j=1}^{n} EAD_j}\sum_{i=1}^{n}\left(EAD_i \cdot \left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} - \mathbb{E}\left[\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}|\tilde{x}\right]\right)\right) \\
&= \sum_{i=1}^{n}\left(w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} - \mathbb{E}\left[w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}|\tilde{x}\right]\right) \\
&= \sum_{i=1}^{n}\left(\tilde{L}_i - \mathbb{E}\left[\tilde{L}_i|\tilde{x}\right]\right) \\
&= \tilde{L} - \mathbb{E}\left[\tilde{L}|\tilde{x}\right]
\end{aligned} \tag{2.134}$$

[120]Negative LGDs are permitted to allow short positions.

this leads to

$$\mathbb{P}\Big(\lim_{n\to\infty}\big(\tilde{L}-\mathbb{E}\big[\tilde{L}|\tilde{x}\big]\big)=0|\tilde{x}=x\Big)=1 \quad \forall x\in\mathbb{R}. \tag{2.135}$$

Using (2.135) it can be shown that the almost sure convergence is also true in the unconditional case:

$$\begin{aligned}\mathbb{P}\Big(\lim_{n\to\infty}\big(\tilde{L}-\mathbb{E}\big[\tilde{L}|\tilde{x}\big]\big)=0\Big) &= \int \mathbb{P}\Big(\lim_{n\to\infty}\big(\tilde{L}-\mathbb{E}\big[\tilde{L}|\tilde{x}\big]\big)=0|\tilde{x}=x\Big)d\mathbb{P}(x)\\ &= \int d\mathbb{P}(x)=1.\end{aligned} \tag{2.136}$$

This completes the proof of (2.129).

2.8.8 Application of Kronecker's Lemma

Proposition. *Assumption (2.83) and (2.84) lead to*

$$\lim_{n\to\infty}\sum_{i=1}^{n} {w_i}^2=0. \tag{2.137}$$

Proof. The following proof is based upon Kronecker's Lemma.[121] Let τ_n be a sequence satisfying

$$0<\tau_1\le\tau_2\le \quad \text{and} \quad \lim_{n\to\infty}\tau_n=\infty. \tag{2.138}$$

If

$$\sum_{n=1}^{\infty} z_n<\infty, \tag{2.139}$$

then

$$\lim_{n\to\infty}\frac{1}{\tau_n}\sum_{i=1}^{n}\tau_i\cdot z_i=0. \tag{2.140}$$

[121] See Petrov (1996), p. 209, Lemma 6.11.

With $\tau_n := \left(\sum_{j=1}^{n} EAD_j\right)^2$ the conditions (2.138) for τ_n are fulfilled due to the first granularity assumption (2.83). Using $z_n := \left(\frac{EAD_n}{\sum_{j=1}^{n} EAD_j}\right)^2$, (2.139) is valid due to the second granularity assumption (2.84). Therefore, Kronecker's Lemma can be applied, which leads to

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{\tau_n}\sum_{i=1}^{n}\tau_i\cdot z_i &= \lim_{n\to\infty}\left(\frac{1}{\left(\sum\limits_{j=1}^{n} EAD_j\right)^2}\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{i} EAD_j\right)^2\cdot\left(\frac{EAD_i}{\sum\limits_{j=1}^{i} EAD_j}\right)^2\right]\right)\\
&= \lim_{n\to\infty}\left(\frac{\sum\limits_{i=1}^{n} {EAD_i}^2}{\left(\sum\limits_{j=1}^{n} EAD_j\right)^2}\right)\\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{EAD_i}{\sum\limits_{j=1}^{n} EAD_j}\right)^2\right)\\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} {w_i}^2\right) = 0,
\end{aligned}
\tag{2.141}
$$

which is (2.137).

2.8.9 Identity of the VaR in the ASRF Model

Proposition. *The following equality is true:*

$$VaR_\alpha\left(\mathbb{E}\left[\tilde{L}|\tilde{x}\right]\right) = \mathbb{E}\left(\tilde{L}|\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \tag{2.142}$$

Proof. Using the notation $\mathbb{E}(\tilde{L}|\tilde{x}) =: g\circ\tilde{x}$,[122] with $g(\tilde{x}) = \mathbb{E}(\tilde{L}|\tilde{x})$, and assuming that the conditional expectation is continuously and strictly monotonously decreasing in x, then there exists a unique inverse g^{-1}, which allows the following transformations:[123]

[122] The notation $g\circ\tilde{x}$ means that some function g is composed with $\tilde{x}$.

[123] See Gordy (2003), p. 207 f., for a similar proof.

$$\begin{aligned} & g \circ \tilde{x} \leq g \circ x \\ \Leftrightarrow & \ g^{-1} \circ g \circ \tilde{x} \geq g^{-1} \circ g \circ x \\ \Leftrightarrow & \ \tilde{x} \geq x \end{aligned} \tag{2.143}$$

and

$$\inf\{g \circ x\} = g \circ \sup\{x\}. \tag{2.144}$$

Using the definition of the VaR (2.15) this leads to the proposition:

$$\begin{aligned} VaR_\alpha\left(\mathbb{E}\left[\tilde{L}|\tilde{x}\right]\right) &= VaR_\alpha(g \circ \tilde{x}) \\ &= \inf\{l|\mathbb{P}[g \circ \tilde{x} > l] \leq 1 - \alpha\} \\ &= \inf\{g \circ x|\mathbb{P}[g \circ \tilde{x} > g \circ x] \leq 1 - \alpha\} \\ &= \inf\{g \circ x|\mathbb{P}[\tilde{x} < x] \leq 1 - \alpha\} \\ &= g \circ \sup\{x|\mathbb{P}[\tilde{x} < x] \leq 1 - \alpha\} \\ &= g \circ \inf\{x|\mathbb{P}[\tilde{x} > x] \leq 1 - \alpha\} \\ &= g \circ VaR_{1-\alpha}(\tilde{x}) \\ &= \mathbb{E}\left(\tilde{L}|\tilde{x} = VaR_{1-\alpha}(\tilde{x})\right). \end{aligned} \tag{2.145}$$

2.8.10 Identity of the ES in the ASRF Model

Proposition. *For $n \to \infty$, the ES of the portfolio loss converges to the ES of the conditional expected loss:*

$$\lim_{n\to\infty} ES_\alpha\left(\tilde{L}\right) = ES_\alpha\left(\mathbb{E}\left(\tilde{L}|\tilde{x}\right)\right). \tag{2.146}$$

Proof. If it is assumed that the loss distribution is continuous, the second term of ES definition (2.19) vanishes.[124] Therefore it only has to be shown that

$$\lim_{n\to\infty} \mathbb{E}\left[\tilde{L} \cdot 1_{\{\tilde{L} \geq q_\alpha(\tilde{L})\}}\right] - \mathbb{E}\left[\mathbb{E}\left(\tilde{L}|\tilde{x}\right) \cdot 1_{\mathbb{E}(\tilde{L}|\tilde{x}) \geq q_\alpha\left(\mathbb{E}(\tilde{L}|\tilde{x})\right)}\right] = 0. \tag{2.147}$$

With $\tilde{X} := \tilde{L} - q_\alpha\left(\tilde{L}\right)$ the first term can be written as

[124] Gordy (2003) shows that it is no necessary condition that the loss distribution has to be continuous. If some additional properties, especially regarding the continuity of the conditional expected loss and of the distribution of the systematic factor, are fulfilled in an interval of x that contains $VaR_\alpha(\tilde{x})$, it follows that $\lim_{n\to\infty} \mathbb{P}\left(\tilde{L} \geq q_\alpha\right) = 1 - \alpha$ so that the second term of the ES definition still vanishes. See Gordy (2003), p. 228 f.

$$\begin{aligned}\mathbb{E}\Big[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\Big] &= \mathbb{E}\Big[\big(\tilde{L}-q_\alpha(\tilde{L})\big)\cdot 1_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\Big] + q_\alpha(\tilde{L})\cdot\mathbb{E}\Big[1_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\Big] \\ &= \mathbb{E}\big[\max(\tilde{X},0)\big] + q_\alpha(\tilde{L})\cdot\mathbb{P}\big[\tilde{L}\geq VaR_\alpha(\tilde{L})\big].\end{aligned} \tag{2.148}$$

Using the shorter notation $\mu(\tilde{x}) := \mathbb{E}(\tilde{L}|\tilde{x})$ and with $\tilde{Y} := \mu(\tilde{x}) - \mu(q_{1-\alpha}(\tilde{x}))$ as well as $\mu(q_{1-\alpha}(\tilde{x})) = q_\alpha(\mu(\tilde{x}))$ from (2.90), the second term of (2.147) equals

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{E}(\tilde{L}|\tilde{x})\cdot 1_{\{\mathbb{E}(\tilde{L}|\tilde{x})\geq q_\alpha(\mathbb{E}(\tilde{L}|\tilde{x}))\}}\Big] \\ &\quad= \mathbb{E}\big[\mu(\tilde{x})\cdot 1_{\{\mu(\tilde{x})\geq q_\alpha(\mu(\tilde{x}))\}}\big] \\ &\quad= \mathbb{E}\big[(\mu(\tilde{x})-\mu(q_{1-\alpha}(\tilde{x})))\cdot 1_{\{\mu(\tilde{x})\geq\mu(q_{1-\alpha}(\tilde{x}))\}}\big] + \mu(q_{1-\alpha}(\tilde{x}))\cdot\mathbb{E}\big[1_{\{\mu(\tilde{x})\geq\mu(q_{1-\alpha}(\tilde{x}))\}}\big] \\ &\quad= \mathbb{E}\big[\max(\tilde{Y},0)\big] + \mu(q_{1-\alpha}(\tilde{x}))\cdot\mathbb{P}[\mu(\tilde{x})\geq\mu(q_{1-\alpha}(\tilde{x}))].\end{aligned} \tag{2.149}$$

Thus, (2.147) can be written as

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\Big[\tilde{L}\cdot 1_{\{\tilde{L}\geq q_\alpha(\tilde{L})\}}\Big] - \mathbb{E}\Big[\mathbb{E}(\tilde{L}|\tilde{x})\cdot 1_{\mathbb{E}(\tilde{L}|\tilde{x})\geq q_\alpha(\mathbb{E}(\tilde{L}|\tilde{x}))}\Big] \\ &\quad= \lim_{n\to\infty}\big(\mathbb{E}\big[\max(\tilde{X},0)\big] + q_\alpha(\tilde{L})\cdot\mathbb{P}\big[\tilde{L}\geq VaR_\alpha(\tilde{L})\big]\big) \\ &\qquad - \big(\mathbb{E}\big[\max(\tilde{Y},0)\big] + \mu(q_{1-\alpha}(\tilde{x}))\cdot\mathbb{P}[\mu(\tilde{x})\geq\mu(q_{1-\alpha}(\tilde{x}))]\big) \\ &\quad= \lim_{n\to\infty}\big(\mathbb{E}\big[\max(\tilde{X},0)-\max(\tilde{Y},0)\big] \\ &\qquad + q_\alpha(\tilde{L})\cdot\mathbb{P}\big[\tilde{L}\geq VaR_\alpha(\tilde{L})\big] - \mu(q_{1-\alpha}(\tilde{x}))\cdot\mathbb{P}[\mu(\tilde{x})\geq\mu(q_{1-\alpha}(\tilde{x}))]\big).\end{aligned} \tag{2.150}$$

Using

$$\lim_{n\to\infty} q_\alpha(\tilde{L}) = \mu(q_{1-\alpha}(\tilde{x})) \tag{2.151}$$

from (2.91) and[125]

$$\lim_{n\to\infty}\mathbb{P}\big[\tilde{L}\geq q_\alpha(\tilde{L})\big] = \mathbb{P}\big[\mu(\tilde{L})\geq\mu(q_\alpha(\tilde{L}))\big] = 1-\alpha, \tag{2.152}$$

the last two terms of (2.150) vanish:

[125] Cf. footnote 118.

$$
\begin{aligned}
&\lim_{n\to\infty} \left(q_\alpha\big(\tilde{L}\big) \cdot \mathbb{P}\big[\tilde{L} \geq VaR_\alpha\big(\tilde{L}\big)\big] - \mu(q_{1-\alpha}(\tilde{x})) \cdot \mathbb{P}[\mu(\tilde{x}) \geq \mu(q_{1-\alpha}(\tilde{x}))]\right) \\
&\quad = \lim_{n\to\infty} \big[q_\alpha\big(\tilde{L}\big) - \mu(q_{1-\alpha}(\tilde{x}))\big] \cdot (1-\alpha) \\
&\quad = 0. \qquad (2.153)
\end{aligned}
$$

Additionally, the inequality $-|x-y| \leq \max(x,0) - \max(y,0) \leq |x-y|$ holds $\forall x, y \in \mathbb{R}$. Using this inequality and (2.151), the remaining first term of (2.150) can be evaluated:

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\big[\max\big(\tilde{X},0\big) - \max\big(\tilde{Y},0\big)\big] &\leq \lim_{n\to\infty} \big|\mathbb{E}\big[\tilde{X} - \tilde{Y}\big]\big| \\
&= \lim_{n\to\infty} \big|\mathbb{E}\big[\tilde{L} - q_\alpha\big(\tilde{L}\big) - [\mu(\tilde{x}) - \mu(q_{1-\alpha}(\tilde{x}))]\big]\big| \\
&= \lim_{n\to\infty} \big|\mathbb{E}\big[\tilde{L} - \mu(\tilde{x})\big] - \big[q_\alpha\big(\tilde{L}\big) - \mu(q_{1-\alpha}(\tilde{x}))\big]\big| \\
&= \lim_{n\to\infty} \big|\mathbb{E}\big(\tilde{L}\big) - \mathbb{E}\big(\mathbb{E}\big(\tilde{L}|\tilde{x}\big)\big) - 0\big| \\
&= 0 \qquad (2.154)
\end{aligned}
$$

and

$$
\lim_{n\to\infty} \mathbb{E}\big[\max\big(\tilde{X},0\big) - \max\big(\tilde{Y},0\big)\big] \geq -\lim_{n\to\infty} \big|\mathbb{E}\big[\tilde{X} - \tilde{Y}\big]\big| = 0. \qquad (2.155)
$$

Thus, the first term vanishes, too, which completes the proof of (2.146).

Chapter 3
Concentration Risk in Credit Portfolios and Its Treatment Under Basel II

3.1 Types of Concentration Risk

Concentration risk can be defined as "any single exposure or group of exposures with the potential to produce losses large enough (relative to a bank's capital, total assets, or overall risk level) to threaten a bank's health or ability to maintain its core operations".[126] There are several types of concentration that can incorporate significant risks (see Fig. 3.1).

In a bank's assets there can be concentration risk arising from obligors and from counterparties of trading transactions. Furthermore, there can be concentrations in collateral instruments or protection sellers. *Market risk* can also contain concentrations, e.g. if there are high exposures in a specific currency. Concentration in a bank's liabilities can arise in refinancing instruments or refinancing counterparties and depositors. These concentrations can lead to an increased *liquidity risk*. Moreover, there can be risk concentration in the execution or processing of transactions, e.g. if there is a high degree of dependence on a specific IT-system. This is referred to as *operational concentration risk*.[127] As lending is usually the main activity of a bank, the most important type of risk concentration is *credit risk* concentration.[128] Against this background, this type of concentration risk will be analyzed in-depth in the following. In the literature, it is often distinguished between three types of credit concentration risk:

- Name concentration
- Sector concentration
- Credit contagion

[126]BCBS (2005a), § 770.

[127]Cf. Deutsche Bundesbank (2006), p. 36 f.

[128]Cf. BCBS (2005a), § 771.

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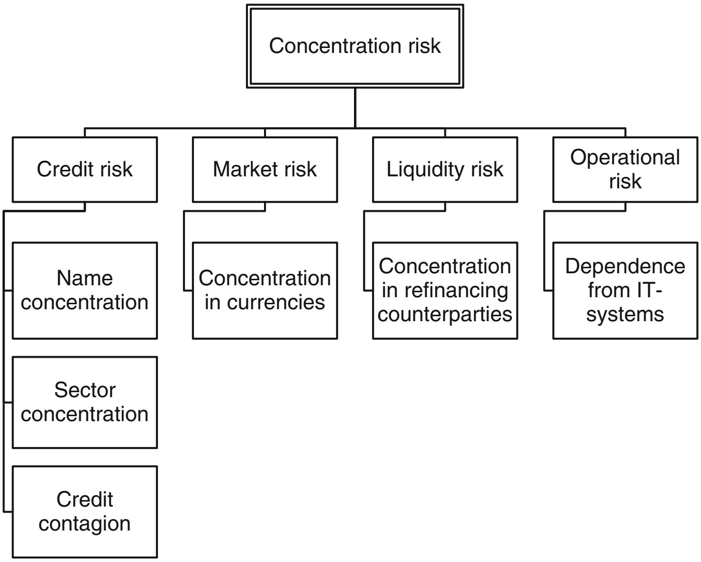

Fig. 3.1 Types of concentration risk. Cf. Deutsche Bundesbank (2006), p. 37

The BCBS distinguishes between two sorts of *name concentration*.[129] One type of concentration risk pertains to an exposure to one firm or to a conglomerate of economically highly dependent firms[130] that is extremely large compared to the rest of the exposures of the portfolio. In such a situation the default risk of the portfolio is significantly driven by the idiosyncratic risk of this individual debtor. This type of concentration will be called "individual name concentration". The second type of name concentration occurs if the bank holds a portfolio containing a relatively small number of firms, each of them with large exposures. Such a portfolio is hardly diversified because of the quite small number of debtors. Thus, a bank faces high losses if several defaults appear, even if they occur accidentally and are not driven by default correlation of the firms. This type of concentration can be denoted as "portfolio name concentration".

The term *sector concentration* refers to significant exposures to groups of counterparts whose likelihood of default is driven by common underlying factors, such as industry sectors or geographical locations.[131] Even if the modeling of these types of sectors is usually similar, the concentrations themselves have some different characteristics. Industry concentrations are mainly related to corporate loans, which have a higher PD if the industry sector is in an economic downturn.

[129]See BCBS (2005b, c).

[130]Under Basel II such a conglomerate is called "connected group"; see BCBS (2005a), § 423.

[131]In a document about technical aspects of the management of concentration risk of the CEBS, examples of common risk factors that possibly lead to sector concentrations also include currencies and credit risk mitigation measures; cf. CEBS (2006), § 25.

In principle, the same is true for geographical concentrations but this type of concentration is also relevant for retail loans and sovereigns. Furthermore, geographical concentration risk includes not only regional but also country risk, which covers different risk categories such as political and transfer risk.[132]

The third type of concentration risk is *credit contagion*. In many cases, two or more companies have a business connection that increases the joint probability of default. This connection is often asymmetric so that a default of firm 1 leads to an increased PD of firm 2 whereas a default of firm 2 shows only minor effects on the PD of firm 1. If the connection is very strong, the firms have to be merged to one connected group. In all other cases only the weaker connection to the overall sector is accounted for, which leads to an underestimation of the true risk.[133] Therefore, credit contagion is in a way between name and sector concentration risk. A typical "micro-structural channel" for this type of concentration risk is the interbank lending market, where a default of one bank could trigger a default of other banks, especially if the loans are uncollateralized and uninsured. Furthermore, suppliers and buyers of goods are often linked via trade credits.[134]

3.2 Incurrence and Relevance of Concentration Risk

Although the expression "concentration risk" expresses the negative aspect of concentration, this does not necessarily mean that it is worthwhile to implement a diversification strategy. As concentration usually stems from *specialization*, a bank can have significant informational advantages in its area of specialization. For example, a bank with a portfolio consisting only of a small number of obligors contains high name concentration but typically knows its obligors very well and can therefore evaluate the firm-specific situation better than others. A bank which is specialized on several industry sectors or geographical locations can have specific knowledge of the relevant markets and the economic environment. As a consequence, in principle a specialized bank could use its informational advantage to generate higher returns and/or lower risk.

In the literature, there exist contradictory statements whether diversification of an intermediary is risk decreasing or increasing and whether diversification increases or decreases the firm value. In *neoclassical economics*, diversification is clearly risk reducing given a constant expected return if the asset returns are

---

[132]Cf. Deutsche Bundesbank (2006), p. 43.

[133]If the connected companies have symmetric dependencies, it would also be possible to build a new sector with a high correlation inside of the sector. However, in practical implementations the sectors are often constructed with respect to geographical regions or industry branches so that the sector factors can be interpreted as macro-economic factors. Hence, the risk stemming from a connection of firms is usually not covered, regardless of whether the connection is symmetric or asymmetric.

[134]Cf. Giesecke and Weber (2006), p. 742.

non-perfectly correlated, which was shown by Markowitz (1952, 1959). Nevertheless, if there do not exist any market imperfections, there is no advantage of a bank's diversification because the diversification could also be done by private investors. Moreover, financial intermediaries would not even exist in the context of the assumed perfect market. An approach which is more suitable for explanation of this strategic decision is the *principal agent theory*. For instance, according to the fundamental work of Diamond (1984), the main task of financial intermediaries is "delegated monitoring" of the obligors, which leads to a reduction of monitoring costs compared to direct investments without an intermediary. Furthermore, the monitoring costs decrease with higher diversification, which directly leads to the result that diversification is advantageous.[135] By contrast, it is often argued that any firm – financial institution or other – should be specialized on a single business line in order to benefit from the management's expertise, whereas diversification should be done by the investors (see Berger and Ofek 1996; Servaes 1996; Denis et al. 1997). In the theoretical model of Winton (1999), several aspects of diversification are addressed: It is assumed that a bank that diversifies into new sectors faces the problem of adverse selection if established banks are already active in the new sectors; this leads to negative consequences on risk and return. Furthermore, monitoring incentives are usually lower when a bank is diversified, leading to a risk-shifting problem. Altogether, even if diversification leads to a smaller impact of downturns in single sectors, it is mostly risk increasing.[136] Empirical studies largely indicate that diversified banks incorporate higher risk and often at the same time lower returns (see Demsetz and Strahan 1997; Acharya et al. 2006; Deng et al. 2007). Furthermore, according to DeLong (2001) only bank mergers which are focused with respect to the dimensions of activity and geography create value. These results are widely in line with the model of Winton (1999).

Relying on the advantages of specialization, the business models of several financial institutions imply a high degree of concentration, like savings banks and credit cooperatives, which are usually regionally focused, and building societies or automotive financial services providers, which are specialized on specific products. Also a combination of both regional and industry expertise is observable, e.g. the HSH Nordbank is the world's largest ship financier but also regionally focused on Germany's North Sea and Baltic Sea coasts and the Stadtsparkasse Köln is a regional savings bank that is specialized on the German media industry.[137]

[135]The monitoring costs are independent of investment size, thus if the monitoring is delegated to an intermediary by many investors, these costs can be reduced. Of course, now the intermediary itself has to be monitored. As state verification is only necessary in case of (imminent) default and the PD of the bank is lower than of the single investments – this is due to diversification of the bank – the monitoring costs of the bank are relatively low leading to the mentioned results.

[136]The model results in a negative effect of diversification on the risk of a bank if the loans in a bank's home sector have high or low PDs. Only in the case of medium default probabilities of the loans in the home sector, diversification can lead to a risk reduction.

[137]Cf. Kamp et al. (2005), p. 1.

The mentioned advantages of specialization are sometimes cited as evidence that there is no additional risk stemming from concentration and therefore there is no additional need for economic capital. Even if some aspects of this argument are comprehensible, it does not hold in general. To begin with, a concentrated bank does not necessarily have informational advantages over other banks. Firstly, it is a necessary condition that the concentration is a result of expertise in the sector. If this is not fulfilled, the concentration is risk increasing only. As mentioned before, in some sectors there might be a multitude of banks with expertise so that the degree of competition highly influences the risk and return. Therefore, a bank must be better in the generation and procession of information than competitors to earn a specialization premium and not to be faced with adverse selection. This point is especially challenging for globally relevant industries as the bank must compete with other financiers worldwide. If a bank has the ability to benefit from specialization, this advantage has to be used not only to increase the return but also to reduce the risk. For example, many venture capital firms or hedge funds have significant industry expertise but do not have a reputation for their risk-averse investments. Moreover, in empirical studies indicating the benefit of specialization, the risk is often measured in terms of volatility.[138] However, as a consequence of non-normality of the portfolio loss distribution, this does not assure that the risk measures which are relevant for economic capital calculations, e.g. the VaR and the ES, are reduced as well. This can be illustrated very intuitively: A bank which is the global market leader in financing of airplanes and of machine tools might be capable of differentiating between risky and less risky lending activities in these areas and uses this knowledge to hold a portfolio with high quality and low volatility. Now assume that one or both of these sectors are faced with a material drop in demand, so there is a sector-specific downturn. Even if the bank perceived some early-warning indicators and was able to reduce the investment in these sectors, there is a high probability that the institution will suffer substantial losses. Thus, in a worst-case scenario, which is relevant for the determination of the economic capital requirement, it is reasonable to assume that many concentrated portfolios are more vulnerable than non-concentrated portfolios. To sum up, there are good arguments that a bank can benefit from specialization in terms of an increased risk/return ratio. But if it has to be assured that the bank survives economic downturn scenarios (with high probability), it should hold an additional capital buffer.

The practical relevance of this issue can be seen from many bank failures or even banking crises that resulted from or at least in combination with concentration risk. During the 1980s and 1990s, more than 1,000 savings and loan associations defaulted in the United States in the savings and loan crisis. Although the problem cannot be reduced to sectoral concentrations, "the banking problems of the 1980s and 1990s came primarily [...] from unsound real estate lending"[139] with a

[138] Cf. Acharya et al. (2006) and Behr et al. (2007).

[139] Seidman et al. (1997), p. 57.

significant increased share of this type of lending.[140] In Scandinavia, the real estate crisis of the early 1990s also led to many bank defaults.[141] The high concentrations in structurally lagging regions led to a high degree of non-performing loans and finally to the divestiture of the Schmidt-Bank in 2001. Also the ongoing financial crisis has its seeds in lax real estate lending, in this case mainly to creditors with low creditworthiness and without down-payment in the United States (subprime lending). A huge amount of the exposure was transferred worldwide to institutional investors via structured financial products, mainly residential mortgage backed securities. These products showed a material price drop, which was due to an underestimation of the correlation between default rates and especially between the residential mortgages. Thus, the concentration risk in the collateral pool was underestimated.[142] In BCBS (2004a) several additional examples of banking crises are studied and a high proportion is found to be affected by risk concentrations.[143]

3.3 Measurement and Management of Concentration Risk

As mentioned in the introductory statement, the Basel Committee on Banking Supervision already recognized the high importance of credit risk concentrations in the Basel framework: *"Risk concentrations are arguably the single most important cause of major problems in banks"*.[144] Against this background, it seems necessary to account for concentration risk in the banks' minimum capital requirements.[145] However, the quantitative framework in Pillar 1 of Basel II does not account for concentration risk at all, since it is based on the ASRF framework, which assumes that (A) the portfolio is infinitely fine-grained and (B) only a single systematic risk factor influences the credit risk of all loans in the portfolio. Thus, the first assumption implies that there is no name concentration in the portfolio, which means that all idiosyncratic risk is diversified completely. The second assumption implicates that there exists no sector concentration such as industry-specific or geographical risk concentrations and also no credit contagion. These are

[140] Prior to the 1980s, less than 10% of U.S. bank portfolios were invested in real estate portfolios, whereas by the mid-1980s some banks increased this share to 50 or 60%; cf. Seidman et al. (1997), p. 58.

[141] Cf. Deutsche Bundesbank (2006), p. 38.

[142] "Structured Finance CDO enhancement levels were not commensurate with the higher observed correlations in the performance of collateral assets during stressed market conditions, particularly for portfolios with elevated risk concentrations or exposure to a narrow, common set of risk factors"; see Hansen et al. (2009), p. 4.

[143] Credit concentration risk, mostly in real estate, is cited to be a relevant cause of bank failures in nine out of the thirteen episodes; cf. BCBS (2004b), p. 66 f.

[144] See BCBS (2005a), § 770.

[145] The term concentration risk will be referred to concentrations in lending if not indicated elsewise.

idealizations that can be problematic for real-world portfolios. But since it is difficult to incorporate credit risk concentrations in analytic approaches and since there is not yet an approach which is widely accepted as the industries "best practice", in Basel II there is no quantitative approach mentioned how to deal with risk concentrations.[146] Instead, it is only qualitatively demanded in Pillar 2 of Basel II that "*Banks should have in place effective internal policies, systems and controls to identify, measure, monitor, and control their credit risk concentrations*".[147] Thus, it is each bank's task to decide how to meet these requirements concretely. However, since this topic is very important for the stability of the banking system, several supervisory documents regarding this issue have been published that analyze the state of the art and give guidance for institutions and supervisors. The Basel Committee launched the "Research Task Force Concentration Risk", which has presented its final report in BCBS (2006). The report contains information about the state of the art in current practice and academic literature, an analysis of the impact of departures from the ASRF model and a review of some methodologies to measure name and sector concentrations. An additional workstream has focused on stress testing against the background of risk concentrations. In 1999 the Joint Forum[148] published "Risk Concentrations Principles" to ensure the prudent management and control of risk concentrations in financial conglomerates through the regulatory and supervisory process. Joint Forum (2008) analyzes the progress of financial conglomerates in identifying, measuring, and managing risk concentrations on a firm-wide basis and across the major risks to which the firm is exposed. Furthermore, the document surveys the current regulatory requirements (quantitative and qualitative) in the European Union, the United States, Japan, and Canada.[149] In CEBS (2006), the Committee of European Banking Supervisors published a survey on current market practices for the identification and measurement of concentration risk. Moreover, five principles for institutions and six principles for supervisors are given as guidance for the treatment of concentration risk under Pillar 2, which specifies the Capital Requirements Directive (CRD) of the European Union regarding concentration risk (see Table 3.1).

[146] Until the second consultative document a version of the so-called granularity adjustment was part of Basel II for measuring name concentrations, but because of some theoretical shortcomings and as it appeared to be too complex for many banks it was cancelled in the final Basel capital rules. The effectiveness and the eligibility of the (cancellation of the) granularity add-on from the second to the third consultative document of Basel II is only discussed vaguely in the literature; see e.g. Bank and Lawrenz (2003), p. 543.

[147] See BCBS (2005a) § 773.

[148] The Joint Forum was established in 1996 under the aegis of the Basel Committee on Banking Supervision (BCBS), the International Organization of Securities Commissions (IOSCO) and the International Association of Insurance Supervisors (IAIS) to deal with issues common to the banking, securities and insurance sectors, including the regulation of financial conglomerates. The Joint Forum is comprised of an equal number of senior bank, insurance and securities supervisors representing each supervisory constituency. See Joint Forum (2009).

[149] Cf. Joint Forum (2008), p. 35 ff.

Table 3.1 Guidance for institutions and supervisors considering concentration risk. See CEBS (2006), p. 11 ff

Guidance for Institutions	
Concentration 1	All institutions should have clear policies and key procedures ultimately approved by the management body in relation to exposure to concentration risk
Concentration 2	In application of Article 22 of the Capital Requirements Directive, institutions should have appropriate internal processes to identify, manage, monitor, and report concentration risk which are suitable to the nature, scale and complexity of their business[a]
Concentration 3	Institutions should use internal limits, thresholds or similar concepts, as appropriate, having regard to their overall risk management and measurement
Concentration 4	Institutions should have adequate arrangements in place for actively monitoring, managing, and mitigating concentration risk against agreed policies and limits, thresholds, or similar concepts
Concentration 5	Institutions should assess the amount of internal capital which they consider to be adequate to hold against the level of concentration risk in their portfolio
Guidance to Supervisors	
Concentration 6	Supervisors will collect sufficient information from institutions on which to base their assessment
Concentration 7	The scope of application of the supervisors' assessment of concentration risk is that used for the Supervisory Review Process (SRP)
Concentration 8	Supervisors will use quantitative indicators, where appropriate, within their Risk Assessment Systems to assess degrees of concentration risk
Concentration 9	The supervisory review should encompass not only quantitative aspects but also the qualitative and organizational aspects of concentration risk management
Concentration 10	Supervisors can draw on stress tests performed by institutions to assess the impact of specific economic scenarios on concentrated portfolios
Concentration 11	Supervisors will pay particular attention to those institutions which are highly concentrated by customer type or specialized nature of product

[a]The Article 22 of the CRD says that every credit institution requires "effective processes to identify, manage, monitor, and report the risks it is or might be exposed to"

As can be seen from these principles, there is a variety of actions that should be taken to handle concentration risk. Due to *Principle* 1, the management body of a credit institution shall define clear policies and procedures regarding concentration risk, which may depend on the business strategy and the risk appetite of the bank. Furthermore, banks should identify and measure concentration risk, which is a necessary condition for managing and monitoring these risks (*Principle* 2). The identification and measurement of concentration risk can be based on rather heuristic or analytical approaches. For example, a bank could measure the size of the top "x" largest exposures or connected exposures relative to the relevant numeraire (e.g. the balance sheet amount). This quantification could also be applied to industry sectors, geographical regions, or product lines.

Moreover, the concentration could be quantified with Gini coefficients or the Herfindahl–Hirschmann Index, which will be described in Sect. 3.4. A review of the literature regarding model-based approaches for the measurement of concentration risk will be given in Sect. 3.5.[150] Principles 3–5 can be seen as further requirements regarding the monitoring and management of concentration risk, which is already demanded in Principle 2. One rather simple but effective action for this purpose is to establish an internal limit system, which shall prevent from undesirable high concentrations in large individual or connected exposures, industry sectors, geographical regions, or product lines (*Principle* 3). A starting point for the limit on large individual exposures can be the directive of the European Union, which demands that a large individual exposure may not exceed 25% of a credit institutions own funds.[151] However, the internal limit system should set additional limits, which are in line with the degree of risk taking that is accepted by the management body. These limits can be based on the aforementioned measurement techniques. Of course, a bank should also have arrangements in place, which actions shall be taken if risk concentrations are detected that are problematic concerning the risk policy or limit system (*Principle* 4). These actions will usually start with a more detailed review of the concerned exposure. Furthermore, stress tests and scenario analyses can be applied. Depending on the results of the analyses, several mitigating actions can be applied, which range from rather passive to active actions. Possible consequences are the modification of concentration limits, the allocation of additional internal capital, a transfer of credit risk to third parties, e.g. using credit derivatives,[152] collateral, or guarantees, and an adjustment of new business acquisitions in order to revert to a lower concentration level. Regardless of whether risk concentrations were originally intended by the bank or not (as it may be the case mentioned in Principle 4), the bank should assess an adequate amount of internal capital against their risk concentrations, which depends on the degree of concentration risk (*Principle* 5). In this context, the onus to demonstrate the adequacy of internal capital will usually be greater for institutions with more concentrated credit portfolios (see also Principle 11). *Principles* 6–11 describe a general guideline for supervisors and advise which actions should be taken during the Supervisory Review Process under Pillar 2 regarding concentration risks.[153] Especially, institutions will be required by supervisors “to show that their internal capital, where considered necessary, is commensurate with the level of concentration risk.”[154] This requirement illustrates that from a regulatory perspective the most important issue is the adequate assessment of capital required in Principle 5.

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[150] Some additional suggestions are given in CEBS (2006).

[151] Cf. EU (2006), Title 5, Chap. 2, Sect. 5, Article 111.1 [Directive 2006/48/EC].

[152] Large exposures will typically be transferred with credit default swaps (CDS), whereas concentrations in sectors or product lines will often be reduced with collateralized debt obligations (CDO). For a short introduction to CDS, especially regarding modeling purposes, see Bluhm et al. (2003). A description of CDOs and analyses of CDOs against the background of asymmetric information between protection seller and protection buyer are given in Gürtler et al. (2008b), whereas Bluhm et al. (2003) as well as Bluhm and Overbeck (2007) present a good overview for modeling CDOs.

[153] Cf. CEBS (2006).

[154] CEBS (2006), p. 2.

The basis for a meaningful monitoring and management of concentration risk is the proper measurement of these risks (for establishing a limit system, for deciding on the quantity of credit derivative instruments, for allocation of internal capital and so on). Against this background, the focus of this work will be the measurement of concentration risk as well as the resulting assessment of the required capital amount. When measuring concentration risk, it is important to notice the popular different interpretations of concentration risk by banks and supervisors. While this is generally unproblematic for internal policies, it is essential for the allocation of additional capital against concentration risk. Banks often only look at one side of concentration risk – the diversification effect. They argue that the Pillar 1 capital requirement does not measure benefits from diversification. Therefore it is argued that this framework is the non-diversified benchmark and thus the upper barrier for the true capital requirement. On the contrary, supervisors interpret concentration risk as "a positive or negative deviation from Pillar 1 minimum capital requirements derived by a framework that does not account explicitly for concentration risk".[155] The latter perception is justified by the fact that the Pillar 1 capital rules were calibrated on well-diversified portfolios with low name and low sector concentration risk.[156] Thus, if a portfolio is lowly diversified, the risk will be underestimated when using the Basel formula. Therefore additional capital is required to

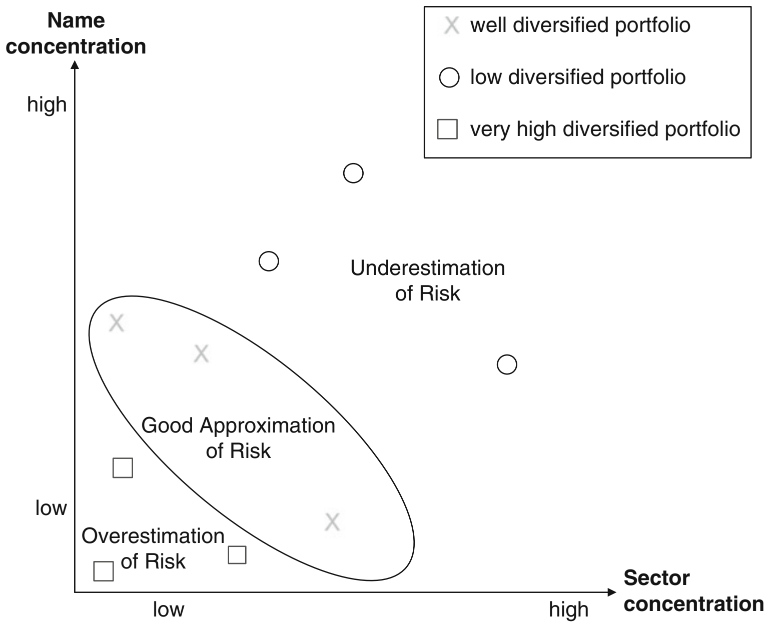

Fig. 3.2 Accuracy of the Pillar 1 capital requirements considering risk concentrations

[155] See BCBS (2006), p. 7.

[156] Cf. BCBS (2006), p.14, and CEBS (2006), § 18.

capture these types of concentration risk. However, if the portfolio is very highly diversified, the Basel formula can overestimate the "true" risk. For well-diversified portfolios, the Basel formula is a good approximation of the "true" risk. This relation is highlighted in Fig. 3.2.

As noticed above, for a quantification of concentration risk there exist some heuristic and some analytical approaches in the literature. Both will be presented in the following sections.

3.4 Heuristic Approaches for the Measurement of Concentration Risk

The most common heuristic approaches quantify concentration risk with the Gini coefficient or the Herfindahl–Hirschmann Index.[157] In principle, both measures can be applied to name concentrations and sector concentrations as well. For a description of the Gini coefficient it is helpful to introduce the *Lorenz curve* first. The Lorenz curve is a graphical representation of the distribution of a variable z and the degree of inequality of this variable. For discrete variables, the Lorenz curve is the piecewise linear function connecting the points (x_i, y_i) with

$$x_i = \frac{i}{n} \quad \text{and} \quad y_i = \frac{\sum_{j=1}^{i} z_{j:n}}{\sum_{j=1}^{n} z_j}, \tag{3.1}$$

where $z_{j:n}$ is the order statistics of z, so that elements of z are sorted into an increasing order.[158] Thus, y_i is the relative amount of the i smallest elements of z_i, and x_i is the relative amount of included elements. For example one point on the Lorenz curve could show that the smallest 20% elements of a variable account for 5% of the total amount.[159] If the elements are of equal size, the function is simply $y = x$, which is called the "line of perfect equality". The opposite, the "line of perfect inequality" is a situation, where one element accounts for the total amount of the variable so that $y = 0$ for all $x < 1$ and $y = 1$ if $x = 1$. Against the background of name concentrations, the variable z could be identified with credit exposures. Thus, the Lorenz curve shows the cumulative share of exposures for

[157] Cf. Deutsche Bundesbank (2006), p. 40 ff., and BCBS (2006), p. 8 ff.

[158] Cf. Sect. 2.2.4.

[159] A common example for the usage of the Lorenz curve is the concentration analysis of income distributions.

each cumulative share of credits.[160] As the relative share of an exposure is defined by the weight w_i, the expression (3.1) simplifies to

$$x_i = \frac{i}{n} \quad \text{and} \quad y_i = \sum_{j=1}^{i} w_{j:n}. \tag{3.2}$$

Fig. 3.3 exemplarily shows the Lorenz curve for credit exposures. The closer the curve is to the diagonal line, the smaller are inequality and concentration of the exposures.

The Lorenz curve is directly related to the *Gini coefficient*, which expresses the degree of inequality in a single number between 0 (perfect equality) and 1 (perfect inequality). As area A between the diagonal line and the Lorenz curve reflects the degree of inequality, the Gini coefficient G is defined as twice this area so that the area is transformed from $A \in [0, 0.5]$ to $G \in [0, 1]$. The area under the Lorenz curve can be calculated as a sum of trapezoids, leading to a Gini coefficient of

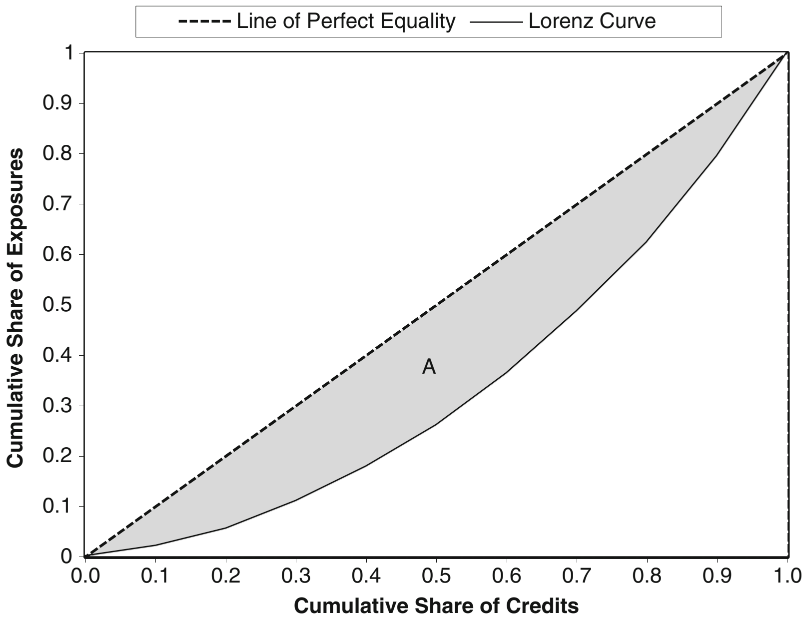

Fig. 3.3 Lorenz curve for credit exposures

[160]In many cases it makes sense to aggregate all credit exposures of one obligor to one exposure before. E.g. in corporate portfolios a default is usually referred to the obligor such that all credits are in default if the obligor is past due more than 90 days on any (material) credit obligation. On the contrary, in retail portfolios the defaults can be handled on contract instead of obligor level. In this case the credits can be handled separately.

$$
\begin{aligned}
G = 2 \cdot A &= 2 \cdot \left(\frac{1}{2} - \sum_{i=1}^{n} \text{Trapezoid}_i \right) \\
&= 2 \cdot \left(\frac{1}{2} - \sum_{i=1}^{n} \frac{1}{2} \cdot (x_i - x_{i-1}) \cdot (y_i + y_{i-1}) \right) \\
&= 1 - \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot (y_i + y_{i-1}).
\end{aligned}
\tag{3.3}
$$

The advantage of the Lorenz curve and the Gini coefficient is that they can easily be implemented and interpreted. However, there are several disadvantages that delimit the benefit. One problem is that the results do not account for the number of credits and therefore for no portfolio name concentration. For example, a poorly diversified portfolio consisting of two credits with exposure weights $w_1 = 0.3$ and $w_2 = 0.7$ has a Gini coefficient of $G = 1 - (0.5 \cdot 0.3 + 0.5 \cdot 0.7) = 0.5$ and the corresponding Lorenz curve is defined by x_i and y_i from (3.2). A portfolio with significantly lower name concentration could be constructed by dividing each of the credits in 100 credits with equal weight, but this portfolio still has the identical Lorenz curve and a Gini coefficient of $G = 0.5$ since the degree of inequality remains identically. Thus, only individual name concentration can be expressed by this method but no portfolio name concentration. Another problem is that no correlation effects and no different portfolio qualities can be accounted for. Two portfolios with identical exposure distributions but different correlation or PD structures have the same Lorenz curve but different name concentrations.

The Lorenz curve and the Gini coefficient can also be applied to sector concentrations. For this purpose, the exposures of each industry sector or each geographical region could be aggregated so that the concentration regarding the exposure size of sectors is measured. The problem that the number of sectors is not accounted for is less problematic because the number of sectors is usually fixed for a single bank. Even if the Lorenz curve is not comparable between different banks due to a different sector definition, the variation of the Lorenz curve in time can be observed for a single bank. However, the problem regarding correlation effects is very critical for sector concentrations, as there is no chance to distinguish between a "diversification" across highly dependent or marginally related sectors.

The *Herfindahl–Hirschmann Index* (HHI) is another measure, which is often used for a quantification of concentrations. As already mentioned in Sect. 2.6, the HHI is defined as the sum of squared weights of elements (exposures) and the reciprocal is the effective number of elements (exposures):

$$
HHI = \sum_{i=1}^{n} {w_i}^2 = \frac{1}{n^*}. \tag{3.4}
$$

In comparison to the Gini coefficient, the advantage of the HHI is that the index accounts for the number of credits, which is relevant for portfolio name concentration. In the example above, the HHI is 0.58 for the two-credit-case and 0.0058 for the case of dividing each of these credits into 100 equal sized credits. Moreover, there is a weak theoretical link between the HHI and name concentration risk as a HHI of zero is a necessary condition for infinite granularity.[161] Hence, the HHI seems to be a better measure of name concentration than the Gini coefficient. As an ad-hoc measure of sector concentration the HHI faces the problems of neglecting the correlation and PD structure, too. Thus, this index can only provide a superficial estimate of sectoral concentrations.

Against this background, the mentioned heuristic approaches should only be used for a rough impression of the degree of concentration in the portfolio and of the variation of concentration in time. Since none of the methods is capable of including correlation effects, which are a core element of concentration risk, and no information about the capital requirement can be achieved, it appears necessary to additionally measure concentration risk with more sophisticated, model-based approaches.

3.5 Review of the Literature on Model-Based Approaches of Concentration Risk Measurement

As noticed in Sect. 3.2, *name concentrations* can be divided into individual name concentrations and portfolio name concentrations. The latter type of name concentrations can be analytically approximated with the so-called granularity adjustment. The idea of the adjustment is based on Gordy (2001), who finds that the add-on for undiversified risk is almost linear in terms of the reciprocal of the number of credits $1/n$ and estimates the slope of the term by simulation based on the CreditRisk$^+$ model. Wilde (2001) derives the granularity adjustment formula analytically by linear approximations around the VaR resulting from the ASRF model. He shows that the formula implemented in the second consultative paper (CP2) of Basel II only leads to convincing results in a CreditRisk$^+$ model but differs from the theoretically derived results when the adjustment formula is calibrated consistent with the Vasicek model. The adjustment formula has been improved by Pykhtin and Dev (2002) so that it is valid for a broader range of PDs. Gordy (2003) generalizes the adjustment formula and numerically analyzes the accuracy of the granularity adjustment when applied to the CreditRisk$^+$ model for several portfolios. Martin and Wilde (2002), Rau-Bredow (2002) and Gordy (2004) obtain the granularity adjustment using a more straightforward approach on the basis of a Taylor series expansion, applying the results of Gouriéroux et al. (2000) for the first two derivatives of the VaR. Using higher derivatives of VaR derived by Wilde (2003), Gürtler et al. (2008a) extend the adjustment to terms of

[161]Cf. (2.86).

higher order to improve the accuracy. Furthermore, they numerically analyze the impact of unsystematic credit risk and the accuracy of the granularity adjustment when applied to the Vasicek model in detail. While these articles use the VaR as the risk measure, Pykhtin (2004) and Rau-Bredow (2005) derive the granularity adjustment for the case of ES being the relevant risk measure. An approach related to Wilde (2001) is the granularity adjustment from Gordy and Lütkebohmert (2007). Their formulas need less data than the original granularity adjustment but are based on the CreditRisk$^+$ model and not on the Vasicek model, which the IRB Approach is based on. In contrast to these approaches, Emmer and Tasche (2005) refer to individual name concentrations. They assume that one single obligor accounts for a significant share of the overall portfolio, while the rest of the portfolio remains infinitely granular. That is why it is called a semi-asymptotic approach.

There also exist a few analytic and semi-analytic approaches that account for *sector concentrations*. One rigorous analytical approach is Pykhtin (2004), which is based on a similar principle as in Martin and Wilde (2002), expanding the Taylor series expansion to a multi-factor context. This multi-factor adjustment is applied to both the VaR and the ES. An alternative is the semi-analytic model from Cespedes et al. (2006). The authors determine a formula that transforms the VaR of the IRB Approach into a multi-factor approximation of the VaR through a complex numerical mapping procedure. Düllmann (2006) extends the binomial extension technique (BET) model from Moody's by incorporating the "infection probability" of Davis and Lo (2001). This additional parameter has been calibrated in a way that the VaR of a multi-factor model is approximated. Based on the principles of Emmer and Tasche (2005), Tasche (2006b) suggests an extension of the ASRF framework towards an asymptotic multi-risk-factor setting. Some numerical work on the performance of the Pykhtin model has been done by Düllmann and Masschelein (2007). Furthermore, Düllmann (2007) presents a first comparison of different approaches on sector concentration risk. Gürtler et al. (2010) adjust the models of Pykhtin (2004) and Cespedes et al. (2006) to be consistent with the IRB Approach. Furthermore, they compare the performance of the models on the basis of a simulation study.

One of the first contributions to the literature that models *credit contagion* is Davis and Lo (2001). In their model, the authors distinguish between direct defaults and indirect defaults, which occur through an infection from directly defaulting firms. Hammarlid (2004) shows how independent sectors can be aggregated within the model of Davis and Lo (2001). Giesecke and Weber (2006) model the probability of financial distress depending on the number of financially distressed business partners in a reduced-form model. However, these contributions assume homogeneous credits – for Hammarlid (2004) at least inside the independent sectors – and a symmetric dependence structure. Neu and Kühn (2004) and Egloff et al. (2007) allow for more realistic credit portfolios consisting of credits with heterogeneous characteristics and asymmetric dependence structures but the computation of loss distributions needs Monte Carlo simulations. Neu and Kühn (2004) is based on a multi-factor default-mode model. The authors add a term to the individual asset return that leads to an increased PD if connected firms are in financial distress and to a decreased PD if competitors default. Egloff et al. (2007) extend a multi-factor model, which allows for rating migrations, with asymmetric microstructural dependencies. In contrast to Neu and Kühn (2004), there is no additional term in the asset return but the idiosyncratic component is divided into a “true” unsystematic fraction and a fraction that is influenced by defaults of related firms.

Chapter 4
Model-Based Measurement of Name Concentration Risk in Credit Portfolios

4.1 Fundamentals and Research Questions on Name Concentration Risk

As described in Sect. 2.6, name concentration risk arises if the idiosyncratic risk cannot be diversified away, which concurrently means that assumption (A) of the ASRF model, the infinite granularity, does not hold. However, a violation of (A) does not have to lead to the fact that the ASRF framework cannot be used at all for credit risk quantification. Nonetheless, the consequences of the violation have to be considered, i.e. the existence of name concentration risk. This issue is not only a problem that should be accounted for in credit risk management when dealing with analytical models, but it is also critical for supervisory capital measurement in banks.[162] This raises the following question: Does assumption (A) of the IRB-model under Pillar 1 generally hold for our portfolio or do we have to quantify name concentration risk for Pillar 2?

Emmer and Tasche (2005) show that the underestimation of *individual name concentrations* can have a significant impact, especially if the exposure weight of a single credit is higher than 2%. Due to the limits on large exposures in the European Union, the exposure to a client may not exceed 25% of a credit institution's own funds.[163] Consequently, a weight of 2% (of total funds) can only be exceeded if (1) more than 8% of a credit institution's capital are own funds and (2) the large exposure limit is reached. This shows that idiosyncratic name concentrations usually should not be problematic if the large exposure rules are effective. Similarly it could be quantified whether *portfolio name concentration* has a significant impact on the risk of the portfolio. In this context, it would be interesting to know which characteristics a real-world bank portfolio should fulfill in order to get a sufficient approximation

[162] Another solution to the problem of the violation of assumption (A) or (B) might be to cancel risk quantification under the IRB Approach and use internal models. However, this solution is not designated in Basel II.

[163] Cf. Sect. 3.2.

M. Hibbeln, *Risk Management in Credit Portfolios*, Contributions to Economics,
DOI 10.1007/978-3-7908-2607-4_4,

of the "true" risk even if name concentrations are not explicitly measured. These characteristics should be determined in a way that the accuracy of the ASRF framework can be easily assessed for a broad range of credit portfolios. If the desired accuracy cannot be achieved using the ASRF model, the VaR of the portfolio could be approximated using the granularity adjustment formula. However, since this formula does not provide an exact solution but an approximation of the risk stemming from portfolio name concentration, it is important to know for which types of credit portfolios the adjustment formula shows an adequate performance. Unfortunately, the existing literature concerning name concentration risk does not answer these questions sufficiently.[164] Against this background, the following important tasks regarding name concentrations will be analyzed in this chapter:

- In which cases are the assumptions of the ASRF framework model critical concerning the credit portfolio size?
- In which cases are currently discussed adjustments for the VaR-measurement able to overcome the shortcomings of the ASRF model?

The answers to both questions would be available if the minimum number of loans, which is necessary to fulfill the granularity assumption (A) with a required accuracy, were known. For this purpose, it could be demanded that the analytically determined VaR and the true VaR using the binomial model of Vasicek shall differ at maximum 5%.[165] Against this background, firstly, the formulas for the (first-order) granularity adjustment will be derived.[166] As the granularity adjustment itself is an asymptotic result, it can be seen as an approximation for medium grained portfolios. Thus, the existent framework will be extended in form of a second-order granularity adjustment in order to account for small sized portfolios.[167] The possibility of such an extension was already mentioned by Gordy (2004) but neither derived nor tested

[164]Gordy (2003) comes to the conclusion that the granularity adjustment works fine for risk buckets of more than 200 loans considering low credit quality buckets and for more than 1,000 loans for high credit quality buckets. However, he uses the CreditRisk^{+} framework from Credit Suisse Financial Products (1997) and not the Vasicek model that builds the basis of Basel II, and he does not analyze the effect of different correlation factors as they are assumed in Basel II.

[165]This question is also interesting when analyzing the Basel II formula because the designated add-on factor for the potential violation of assumption (A) was cancelled from the second consultative document to the third consultative document; see BCBS (2001a, 2003a). Thus, we only prove under which conditions the assumption (A) of the Vasicek model is fulfilled. Of course, this model may suffer from other assumptions like the distributional assumption of standardized returns. However, since we would only like to address the topic of concentration risk, our focus should be reasonable. Additionally, the distributional assumptions seem not to have a deep impact on the measured VaR; see Koyluoglu and Hickman (1998a, b), Gordy (2000) or Hamerle and Rösch (2005a, b, 2006).

[166]Wilde (2001) calls this "the granularity adjustment to first order in the unsystematic variance".

[167]This procedure can be motivated by the fact that for market risk quantification of nonlinear exposures two factors of the Taylor series (fist and second order) are common to achieve a higher accuracy; see e.g. Crouhy et al. (2001) or Jorion (2001). This might be appropriate for credit risk as well. Furthermore, the higher order derivatives of VaR given by Wilde (2003) make it possible to systematically derive such a formula.

so far. Secondly, the minimum number of loans in a portfolio will be inferred numerically using two definitions of accuracy in order to enhance the theoretical background with concrete facts on critical portfolio sizes.[168] This could give an advice which sub-portfolios have significant risk concentrations and thus should be controlled on credit portfolio and not on individual credit level. In the first analyses it will be focused on homogeneous credit portfolios, i.e. each borrower has an identical PD as well as an identical EAD and LGD. Furthermore, the granularity adjustment of an inhomogeneous portfolio will be examined on the basis of Monte Carlo simulations as well. These analyses contribute to the explanation of differences between simulated and analytically determined solutions to credit portfolio risk as well as between Basel II capital requirements for Pillar 2 with respect to Pillar 1.[169]

Although it could be shown that the non-coherency of the VaR is not relevant for the ASRF model, this result does not hold anymore if the assumption of infinite granularity is not fulfilled. Thus, in Sect. 4.3 the derivation of the granularity adjustment and the aforementioned numerical analyses will be performed for the ES as well. In addition, the performance of the ES-based granularity adjustment will be tested for portfolios with stochastic LGDs. Beside the theoretical advantages of the ES, the results of the numerical study demonstrate that the granularity adjustment generates better approximations for the ES than for the VaR. Moreover, even if stochastic LGDs are included as an additional source of uncertainty, the accuracy of the adjustment formula is very high.

4.2 Measurement of Name Concentration Using the Risk Measure Value at Risk[170]

4.2.1 Considering Name Concentration with the Granularity Adjustment

4.2.1.1 First-Order Granularity Adjustment for One-Factor Models

The principle of incorporating the effect of the portfolio size in a one-factor model is very simple. As a first step, it is assumed that the portfolio is infinitely fine

[168]The Basel Committee on Banking Supervision already stated that in principle the effect of portfolio size on credit risk is well understood but lacks practical analyses; see BCBS (2005b).

[169]Additionally, this study makes contribution to the ongoing research on analyzing differences between Basel II capital requirements and banks internal “true” risk capital measurement approaches. Since the harmonization of the regulatory capital requirements and the perceived risk capital of banks internal estimates for portfolio credit risk is often stated as the major benefit of Basel II, see e.g. Hahn (2005), p. 127, but often not observed in practice, this task might be of relevance in the future.

[170]The main results of this section comply with Gürtler et al. (2008a).

grained and the VaR can be determined under the ASRF framework. However, an add-on factor is constructed, which accounts for the finite size of the portfolio and converges to zero if assumption (A) of infinite granularity is (nearly) met. This factor can be determined in form of the first element different from zero that results from a Taylor series expansion of the VaR around the ASRF solution. An alternative approach is to evaluate the unintentional shift of the confidence level due to the negligence of granularity and to transform the result into a shift of the loss quantile. The approximation is based on some linearizations around the systematic loss. Hence, both approaches rely on the proximity of the true VaR and the VaR under the ASRF framework. As the implementation of the Taylor series expansion is more straightforward, the following explanations are referred to this approach. The pioneer work on the granularity adjustment of Wilde (2001), which relies on the other approach mentioned, is presented in Appendix 4.5.1.

In order to perform the Taylor series expansion, the portfolio loss will be subdivided into a systematic and an unsystematic part, i.e.

$$\tilde{L} = \mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right) + \left[\tilde{L} - \mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)\right] =: \tilde{Y} + \lambda\tilde{Z}. \tag{4.1}$$

Thus, the first term $\mathbb{E}(\tilde{L}\,|\,\tilde{x}) =: \tilde{Y}$ describes the systematic part of the portfolio loss that can be expressed as the expected loss conditional on $\tilde{x}$ (see also (2.85)). The second term $\tilde{L} - \mathbb{E}(\tilde{L}\,|\,\tilde{x}) =: \lambda\tilde{Z}$ of (4.1) stands for the unsystematic part of the portfolio loss, which results from the idiosyncratic risk. Therefore, $\tilde{Z}$ describes the general idiosyncratic component and λ decides on the fraction of the idiosyncratic risk that stays in the portfolio. Obviously, λ tends to zero if the number of obligors n converges to infinity, since this fraction (of the idiosyncratic risk) vanishes if granularity assumption (A) from Sect. 2.6 holds. However, for a granularity adjustment we claim that the portfolio is only "nearly" infinitely granular and thus λ is just close to but exceeds zero. In order to incorporate the idiosyncratic part of the portfolio loss into the VaR-formula, we perform a *Taylor series expansion around the systematic loss* at $\lambda = 0$. We get

$$\begin{aligned} VaR_\alpha\left(\tilde{L}\right) &= VaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right) \\ &= VaR_\alpha\left(\tilde{Y}\right) + \lambda\left[\frac{dVaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2!}\left[\frac{d^2VaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda^2}\right]_{\lambda=0} \\ &\quad + \cdots + \frac{\lambda^m}{m!}\left[\frac{d^mVaR_\alpha\left(\tilde{Y} + \lambda\tilde{Z}\right)}{d\lambda^m}\right]_{\lambda=0} + \cdots \quad . \end{aligned} \tag{4.2}$$

Thus, the first term describes the systematic part of the VaR and all other terms add an additional fraction to the VaR due to the undiversified idiosyncratic component. If the Taylor series expansion is formed up to the quadratic term, the first two derivatives of VaR are needed. According to Gouriéroux et al. (2000), the *first and second derivative of VaR* are given as[171]

$$\left.\frac{dVaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda}\right|_{\lambda=0} = \mathbb{E}\left[\tilde{Z}|\tilde{Y}=q_\alpha\left(\tilde{Y}\right)\right], \tag{4.3}$$

$$\left.\frac{d^2VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda}\right|_{\lambda=0} = -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}, \tag{4.4}$$

with $f_Y(y)$ being the probability density function of $\tilde{Y}$. Concurrently, the first derivative of VaR equals zero[172]:

$$\mathbb{E}\left(\tilde{Z}\,|\,\tilde{Y}\right) = \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}-\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)|\,\tilde{Y}\right) = \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}\,|\,\tilde{Y}\right) - \frac{1}{\lambda}\cdot\mathbb{E}\left(\tilde{L}\,|\,\tilde{Y}\right) = 0, \tag{4.5}$$

so that the second derivative is the first relevant element underlying the granularity adjustment. With

$$\lambda^2\cdot\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\lambda\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}-\tilde{Y}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}\right], \tag{4.6}$$

the quadratic term of the Taylor series expansion (4.2) results in

$$\begin{aligned}\Delta l_1 &= \frac{\lambda^2}{2}\left(-\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}\right)\\ &= -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\,\mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}.\end{aligned} \tag{4.7}$$

As the conditional expectation $\tilde{Y}=\mathbb{E}(\tilde{L}\,|\,\tilde{x})$ is continuous and strictly monotonously decreasing in x, the probability density function $f_Y(y)$ can be transformed into[173]

$$f_Y(y) = \frac{f_x(x)}{|dy/dx|} = -\frac{f_x(x)}{dy/dx} = -\frac{f_x(x)}{\frac{d}{dx}\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}=x\right)}. \tag{4.8}$$

[171] See Appendix 4.5.2.

[172] This is valid because the added risk of the portfolio is unsystematic; see Martin and Wilde (2002) for further explanations.

[173] See Appendix 4.5.3.

Furthermore, using (4.8) and[174]

$$\begin{aligned} &\tilde{Y} = q_\alpha(\tilde{Y}) \\ &\quad \Leftrightarrow \mathbb{E}(\tilde{L}\,|\,\tilde{x}) = q_\alpha(\mathbb{E}(\tilde{L}\,|\,\tilde{x})) \\ &\quad \Leftrightarrow \mathbb{E}(\tilde{L}\,|\,\tilde{x}) = \mathbb{E}(\tilde{L}\,|\,q_{1-\alpha}(\tilde{x})) \\ &\quad \Leftrightarrow \tilde{x} = q_{1-\alpha}(\tilde{x}), \end{aligned} \tag{4.9}$$

the true quantile of a granular portfolio $VaR_\alpha^{(n)}$ can be approximated by the Taylor series expansion up to the quadratic term, which leads to the following formula for the VaR including the *granularity adjustment* Δl_1:

$$\begin{aligned} &VaR_\alpha^{(n)} \approx VaR_\alpha^{(\mathrm{ASRF})} + \Delta l_1 =: VaR_\alpha^{(\text{1st Order Adj.})} \\ &\text{with } \Delta l_1 = -\frac{1}{2f_x(x)}\frac{d}{dx}\left(\frac{f_x(x)\mathbb{V}[\tilde{L}\,|\,\tilde{x}=x]}{\frac{d}{dx}\mathbb{E}[\tilde{L}\,|\,\tilde{x}=x]}\right)\Bigg|_{x=q_{1-\alpha}(\tilde{x})}. \end{aligned} \tag{4.10}$$

This corresponds to the result of Wilde (2001) and Rau-Bredow (2002). Thus, the VaR figure of the infinitely fine grained portfolio is adjusted by an additional term, that is the first term different from zero of the Taylor series expansion (4.2). In contrast to the ASRF solution, which relies on the conditional expectation only, the granularity adjustment takes the conditional variance of the portfolio loss into account. In the following, the expression above will be called the ASRF solution with first-order (granularity) adjustment.

A more detailed analysis of (4.10) will show that the granularity adjustment is a term of order $O(1/n^*)$, or for homogeneous portfolios simply $O(1/n)$.[175] For this purpose, the conditional expectation and variance will be looked at. Due to the conditional independence of the credit events and due to the restriction of the individual loss rate $(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}})$ to $[-1,\ 1]$ for all $i \in \{1,\ \ldots,\ n\}$, there exists a finite number $V^*(x) \leq 1$ such that

$$\begin{aligned} \mathbb{V}(\tilde{L}\,|\,\tilde{x}=x) &= \mathbb{V}\left(\sum_{i=1}^n w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x}=x\right) = \sum_{i=1}^n {w_i}^2 \cdot \mathbb{V}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x}=x\right) \\ &= \sum_{i=1}^n {w_i}^2 \cdot V^*(x) = V^*(x) \cdot \sum_{i=1}^n {w_i}^2 = V^*(x) \cdot \frac{1}{n^*}. \end{aligned} \tag{4.11}$$

[174] Cf. the identity 2.90.

[175] The notation n^* refers to the effective number of credits as introduced in (2.87).

Under the same conditions there also exists a finite number $E^*(x) \leq 1$ such that

$$\begin{aligned}\mathbb{E}\left(\tilde{L} \mid \tilde{x} = x\right) &= \mathbb{E}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) = \sum_{i=1}^{n} w_i \cdot \mathbb{E}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) \\ &= E^*(x) \cdot \sum_{i=1}^{n} w_i = E^*(x). \end{aligned} \tag{4.12}$$

Using these expressions, the granularity add-on Δl_1 from (4.10) can be written as

$$\Delta l_1 = -\frac{1}{n^*} \frac{1}{2 f_x(x)} \frac{d}{dx}\left(\frac{f_x(x) V^*(x)}{\frac{d}{dx} E^*(x)}\right)\Bigg|_{x = q_{1-\alpha}(\tilde{x})} = O\left(\frac{1}{n^*}\right). \tag{4.13}$$

This shows that the granularity adjustment is linear in terms of $1/n^*$, so that in a homogeneous portfolio the add-on for undiversified idiosyncratic risk is halved if the number of credits is doubled. This corresponds to the heuristic approach of Gordy (2001), who presumed that the add-on is constant in terms of $1/n$ and estimated the slope of this term by simulation. At the same time it has to be stated that neglecting the additional terms of the Taylor series expansion, which are at least of order $O(1/n^2)$ in the homogeneous case,[176] implies that all higher moments like the conditional skewness and kurtosis are ignored. This can be made clear by expressing the higher conditional moments about the mean η_m similar to (4.11) and (4.12) as[177]

$$\begin{aligned}\eta_m\left(\tilde{L} \mid \tilde{x} = x\right) &= \sum_{i=1}^{n} {w_i}^m \cdot \eta_m\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} | \tilde{x} = x\right) = {\eta_m}^*(x) \cdot \sum_{i=1}^{n} {w_i}^m \\ &\leq {\eta_m}^*(x) \cdot \sum_{i=1}^{n} \left(\frac{b}{n \cdot a}\right)^m = {\eta_m}^*(x) \cdot \left(\frac{b}{a}\right)^m \cdot \frac{1}{n^{m-1}} \\ &= O\left(\frac{1}{n^{m-1}}\right), \end{aligned} \tag{4.14}$$

with some finite numbers ${\eta_m}^*(x) \leq 1$ and $0 < a \leq EAD_i \leq b$ for all i. If higher moments like the conditional skewness shall be considered for the granularity adjustment, too, it would be necessary to include additional elements of the Taylor series expansion. This will be done in the subsequent Sect. 4.2.1.3, but beforehand, the first-order granularity adjustment will be applied to the Vasicek model.

[176] The equivalent term for heterogeneous portfolios is $O\left(\sum_{i=1}^{n} w^3\right)$.

[177] The mth moment of a random variable $\tilde{X}$ about the mean $\eta_m(\tilde{X})$ is defined as $\eta_m(\tilde{X}) := \mathbb{E}([\tilde{X} - \mathbb{E}(\tilde{X})]^m)$; cf. Abramowitz and Stegun (1972), 26.1.6.

4.2.1.2 First-Order Granularity Adjustment for the Vasicek Model

Formula (4.10) is the general result of the granularity adjustment for one-factor models, which could be applied to different models. The application to the one-factor version of CreditRisk$^+$ is demonstrated in Wilde (2001). In the following, the granularity add-on will be specified for the Vasicek model. Thus, the conditional probability of default is assumed to be given by

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.15}$$

and the systematic factor $f_x(x) = \varphi$ is standard normally distributed. For ease of notation, the mth moment of some random variable $\tilde{X}$ about the origin will be denoted by $\mu_m(\tilde{X}) := \mathbb{E}(\tilde{X}^m)$, and the mth conditional moment of the portfolio loss about the origin will be indicated by

$$\mu_{m,c} := \mu_m\left(\tilde{L}\,|\,\tilde{x} = x\right). \tag{4.16}$$

As noticed before, the mth moment of a random variable $\tilde{X}$ about the mean is represented by $\eta_m(\tilde{X}) := \mathbb{E}([\tilde{X} - \mathbb{E}(\tilde{X})]^m)$, and the mth conditional moment of the portfolio loss about the mean will be denoted by

$$\eta_{m,c} := \eta_m\left(\tilde{L}\,|\,\tilde{x} = x\right). \tag{4.17}$$

Using this notation, the conditional expectation and the conditional variance are indicated by $\mu_{1,c}$ and $\eta_{2,c}$, respectively, and the granularity adjustment (4.10) can be expressed as[178]

$$\begin{aligned}\Delta l_1 &= -\frac{1}{2\varphi}\frac{d}{dx}\left(\frac{\varphi\,\eta_{2,c}}{d\mu_{1,c}/dx}\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &= \frac{1}{2}\left[\frac{x\cdot\eta_{2,c}}{d\mu_{1,c}/dx} - \frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx} + \frac{\eta_{2,c}\cdot d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}.\end{aligned} \tag{4.18}$$

Thus, the first and second derivatives of the conditional expectation as well as the first derivative of the conditional variance have to be determined. For this purpose, it will be assumed that the LGDs are stochastically independent of each

[178] Cf. Appendix 4.5.4.

other.[179] Furthermore, the expectation and variance of LGD will be denoted by *ELGD* and *VLGD*, respectively. The required moments are given as[180]

$$\mu_{1,c} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(x), \tag{4.19}$$

$$\eta_{2,c} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right) \cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x)\right]. \tag{4.20}$$

Thus, the needed derivatives are given as

$$\frac{d\mu_{1,c}}{dx} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d(p_i(x))}{dx}, \tag{4.21}$$

$$\frac{d^2\mu_{1,c}}{dx^2} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d^2(p_i(x))}{dx^2}, \tag{4.22}$$

$$\frac{d\eta_{2,c}}{dx} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right) \cdot \frac{d(p_i(x))}{dx} - ELGD_i^2 \cdot \frac{d\left(p_i^2(x)\right)}{dx}\right]. \tag{4.23}$$

According to this, the first two derivatives of $p_i(x)$ as well as the first derivative of ${p_i}^2(x)$ have to be determined. Using the notation

$$p_i(x) = \Phi(z_i), \quad \text{with } z_i = \frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}x}{\sqrt{1-\rho_i}}, \tag{4.24}$$

we obtain

$$\frac{d(p_i(x))}{dx} = \frac{d}{dx}\Phi(z_i) = -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \varphi(z_i), \tag{4.25}$$

$$\frac{d^2(p_i(x))}{dx^2} = -\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \frac{d}{dx}\varphi(z_i) = -\frac{\rho_i}{1-\rho_i} \cdot z_i \cdot \varphi(z_i), \tag{4.26}$$

[179]This assumption can be critical for real-world portfolios. Especially, it is often assumed in ongoing research on credit portfolio modeling that the LGD is dependent on the systematic factor. However, the granularity adjustment formula would complicate significantly as neither the ELGD nor the VLGD could be treated as constant for the derivatives. Against this background, this assumption will be retained for the derivation.

[180]Cf. Appendix 4.5.4. Pykhtin and Dev (2002) corrected the formulas of Wilde (2001), who neglected the last term of the following conditional variance.

$$\frac{d\big(p_i^2(x)\big)}{dx} = \frac{d}{dx}(\Phi(z_i))^2 = -2 \cdot \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \Phi_i(z_i) \cdot \varphi(z_i). \tag{4.27}$$

Formulas (4.21)–(4.27) have to be inserted into (4.18) to get the granularity adjustment. This leads to the following expression for the first-order granularity adjustment for heterogeneous portfolios in the Vasicek model:

$$\begin{aligned}\Delta l_1 = \frac{1}{2}\Bigg[&\Phi^{-1}(\alpha)\frac{\sum_{i=1}^n w_i^2\big[\big(ELGD_i^2+VLGD_i\big)\Phi(z_i) - ELGD_i^2\Phi^2(z_i)\big]}{\sum_{i=1}^n w_i ELGD_i \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\cdot\varphi(z_i)} \\ &-\frac{\sum_{i=1}^n w_i^2\Big[\big(ELGD_i^2+VLGD_i\big)\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i) - 2ELGD_i^2\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\Phi_i(z_i)\varphi(z_i)\Big]}{\sum_{i=1}^n w_i ELGD_i\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i)} \\ &-\sum_{i=1}^n w_i^2\big[\big(ELGD_i^2+VLGD_i\big)\Phi(z_i) - ELGD_i^2\Phi^2(z_i)\big] \\ &\cdot\frac{\sum_{i=1}^n w_i ELGD_i\frac{\rho_i}{1-\rho_i}z_i\varphi(z_i)}{\Big(\sum_{i=1}^n w_i ELGD_i\frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}\varphi(z_i)\Big)^2}\Bigg]_{z_i=\frac{\Phi^{-1}(PD_i)+\sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}}.\end{aligned} \tag{4.28}$$

For homogeneous portfolios, this formula can be simplified to[181]

$$\begin{aligned}\Delta l_1 = \frac{1}{2n}\Bigg(&\frac{ELGD^2+VLGD}{ELGD}\left[\frac{\Phi(z)}{\varphi(z)}\frac{\Phi^{-1}(\alpha)(1-2\rho)+\Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 1\right] \\ &-ELGD\cdot\Phi(z)\left[(z)\frac{\Phi^{-1}(\alpha)(1-2\rho)+\Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 2\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}},\end{aligned} \tag{4.29}$$

which is the formula presented by Pykhtin and Dev (2002).

4.2.1.3 Second-Order Granularity Adjustment for One-Factor Models

Recalling the discussion of the first-order granularity adjustment, the ASRF solution might only lead to good approximations if term (4.28) of order $O(1/n)$ is close

[181] Cf. Appendix 4.5.5.

to zero, whereas the ASRF solution including the first-order granularity adjustment might only be sufficient if the terms of order $O(1/n^2)$ vanish. For medium sized risk buckets this might be true, but if the number of credits in the portfolio is getting considerably small, an additional factor might be appropriate. Particularly, the mentioned granularity adjustment is linear in $1/n$ and this might not hold for small portfolios. Indeed, Gordy (2003) shows by simulation that the portfolio loss seems to follow a concave function and therefore adjustment (4.28) would slightly overshoot the theoretically optimal add-on for smaller portfolios.[182] An explanation of the described behavior is that the first-order adjustment only takes the conditional variance into account whereas higher conditional moments, which result from higher order terms, are ignored. As noticed in Sect. 4.1, additional elements of the Taylor series expansion (4.2) will be calculated in the following with the intention to improve the adjustment for small portfolio sizes. Hence, all elements of order $O(1/n^2)$ will be taken into account, and thus the error will be reduced to $O(1/n^3)$.[183] This newly derived formula will be called the *second-order granularity adjustment*. The resulting ASRF solution including the first and the second-order granularity adjustment Δl_2 is

$$VaR_\alpha^{(\text{1st + 2nd Order Adj.})} = VaR_\alpha^{(\text{ASRF})} + \Delta l_1 + \Delta l_2, \tag{4.30}$$

where Δl_2 represents the $O(1/n^2)$ elements of (4.2).

In order to calculate these elements, higher derivatives of VaR are required. Referring to Wilde (2003), a formula for all derivatives of VaR is derived in Appendix 4.5.6. Having a closer look at the derivatives of VaR, the fourth and a part of the fifth element of the Taylor series are identified to be relevant for the $O(1/n^2)$ terms.[184] Thus, the third and the fourth derivative of VaR are required. As shown in Appendix 4.5.7, the rather complex result for all derivatives can be simplified for the first five derivatives ($m = 1,\ 2,\ \ldots,\ 5$) of VaR to

$$\begin{aligned}\left.\frac{\partial^m VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{\partial\lambda^m}\right|_{\lambda=0} &= (-1)^m\left(-\frac{1}{f_Y(y)}\right)\Bigg[\frac{d^{m-1}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-1}} \\ &\quad -\kappa(m)\cdot\frac{d}{dy}\left(\frac{1}{f_Y(y)}\cdot\frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy}\frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-3}}\right)\Bigg]_{y=q_\alpha\left(\tilde{Y}\right)},\end{aligned} \tag{4.31}$$

with $\kappa(1) = \kappa(2) = 0,\ \kappa(3) = 1,\ \kappa(4) = 3$, and $\kappa(5) = 10$.

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[182] Gordy (2003) observes the concavity of the granularity add-on for a high-quality portfolio (A-rated) up to a portfolio size of 1,000 debtors.

[183] See Gordy (2004), p. 112, footnote 5, for a similar suggestion.

[184] See Appendix 4.5.8 for details regarding the order of these elements.

Using the third and the fourth derivative of VaR and due to[185]

$$\lambda^m \cdot \mu_m\left(\tilde{Z} \,|\, \tilde{Y} = y\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)} = \eta_m\left[\tilde{L} \,|\, \tilde{Y} = y\right]\Big|_{y=q_\alpha\left(\tilde{Y}\right)} =: \eta_m(y)\big|_{y=q_\alpha\left(\tilde{Y}\right)} \tag{4.32}$$

as well as $\eta_1(y) = 0$, the elements of order $O(1/n^2)$ of the Taylor series expansion (4.2) are given as

$$\begin{aligned}
\Delta l_2 &= \frac{(-1)^3}{3!}\left(-\frac{1}{f_Y(y)}\right)\left[\frac{d^2(\eta_3(y)f_Y(y))}{dy^2} - \frac{d}{dy}\left(\frac{1}{f_Y(y)}\frac{d(\eta_2(y)f_Y(y))}{dy}(\eta_1(y)f_Y(y))\right)\right] \\
&\quad + \frac{(-1)^4}{4!}\left(-\frac{1}{f_Y(y)}\right)\left[-3\frac{d}{dy}\left(\frac{1}{f_Y(y)}\frac{d(\eta_2(y)f_Y(y))}{dy}\frac{d(\eta_2(y)f_Y(y))}{dy}\right)\right]\Bigg|_{y=q_\alpha\left(\tilde{Y}\right)} \\
&= \frac{1}{6}\frac{1}{f_Y(y)}\frac{d^2}{dy^2}[\eta_3(y)f_Y(y)] + \frac{1}{24}\frac{3}{f_Y(y)}\frac{d}{dy}\left[\frac{1}{f_Y(y)}\left(\frac{d}{dy}[\eta_2(y)f_Y(y)]\right)^2\right]\Bigg|_{y=q_\alpha\left(\tilde{Y}\right)}.
\end{aligned} \tag{4.33}$$

Recalling that $\mu_{m,c} = \mu_m(\tilde{L} \,|\, \tilde{x} = x)$, $f_Y(y) = -\frac{f_x(x)}{dy/dx}$ (see (4.8)), and $\eta_m(y)|_{y=q_\alpha(\tilde{Y})} := \eta_m(\tilde{L} \,|\, \tilde{Y} = q_\alpha(\tilde{Y})) = \eta_m(\tilde{L} \,|\, \tilde{x} = q_{1-\alpha}(\tilde{x})) =: \eta_{m,c}\big|_{x=q_{1-\alpha}(\tilde{x})}$ (cf. (4.9) and (4.32)), Δl_2 can be written as

$$\begin{aligned}
\Delta l_2 &= \frac{1}{6f_x}\frac{d}{dx}\left(\frac{d}{dy}\left[\frac{\eta_{3,c}f_x}{dy/dx}\right]\right) + \frac{1}{8f_x}\frac{d}{dx}\left[\frac{1}{f_x}\frac{dy}{dx}\left(\frac{d}{dy}\left[\frac{\eta_{2,c}f_x}{dy/dx}\right]\right)^2\right]\Bigg|_{x=q_{1-\alpha}(\tilde{x})} \\
&= \frac{1}{6f_x}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left[\frac{\eta_{3,c}f_x}{d\mu_{1,c}/dx}\right]\right) \\
&\quad + \frac{1}{8f_x}\frac{d}{dx}\left[\frac{1}{f_x}\frac{1}{d\mu_{1,c}/dx}\left(\frac{d}{dx}\left[\frac{\eta_{2,c}f_x}{d\mu_{1,c}/dx}\right]\right)^2\right]\Bigg|_{x=q_{1-\alpha}(\tilde{x})},
\end{aligned} \tag{4.34}$$

which is our general result for the second-order granularity adjustment. Having a closer look at (4.34), it can be seen that the second-order adjustment takes a squared term of the conditional variance as well as the conditional skewness into account,[186] which are both of order $O(1/n^2)$.[187]

[185] Cf. (4.236) of Appendix 4.5.8.

[186] Precisely, the element $\eta_{3,c}$ is the third conditional moment centered about the mean whereas the conditional skewness is the "normalized" third moment, defined as the third conditional moment about the mean divided by the conditional standard deviation to the power of three.

[187] Cf. (4.14).

4.2.1.4 Second-Order Granularity Adjustment for the Vasicek Model

Similar to Sect. 4.2.1.2, we specify our general result of the second-order granularity adjustment for the Vasicek model with

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.35}$$

and a standard normally distributed systematic factor, leading to $f_x = \varphi$ and $q_{1-\alpha}(\tilde{x}) = \Phi^{-1}(1-\alpha)$. As derived in Appendix 4.5.9 under the assumption of a standard normally distributed systematic factor, the second-order granularity adjustment is equivalent to

$$\begin{aligned}
\Delta l_2 = &\frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\left[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right.\\
&\left.+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\right]\\
&+\frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.\\
&+2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.\\
&\left.\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]\right|_{x=\Phi^{-1}(1-\alpha)}.
\end{aligned} \tag{4.36}$$

As can be seen from (4.36), Δl_2 is a function of $\mu_{1,c}$, $\eta_{2,c}$, and $\eta_{3,c}$. According to (4.19), (4.20), and (4.264),[188] these moments are given as

$$\mu_{1,c} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(x), \tag{4.37}$$

$$\eta_{2,c} = \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right)\cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x)\right], \tag{4.38}$$

[188] See Appendix 4.5.10.

$$\eta_{3,c} = \sum_{i=1}^{n} w_i^3 \big[\big(ELGD_i^3 + 3 \cdot ELGD_i \cdot VLGD_i + SLGD_i\big) \cdot p_i(x) \\ -3 \cdot \big(ELGD_i^3 + ELGD_i \cdot VLGD_i\big) \cdot p_i^2(x) + 2 \cdot ELGD_i^3 \cdot p_i^3(x)\big], \quad (4.39)$$

with $SLGD := \eta_3(\widetilde{LGD})$. The conditional PD from (4.35) can be written as

$$p_i(x) = \Phi(z_i), \quad \text{with } z_i = \frac{\Phi^{-1}(PD_i)}{\sqrt{1-\rho_i}} - s_i \cdot x \quad \text{and} \quad s_i = \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}}. \quad (4.40)$$

Using this notation and having a closer look at (4.36) and the conditional moments, we find that the following derivatives are needed

$$\frac{d(p_i(x))}{dx} = -s_i \cdot \varphi(z_i), \quad (4.41)$$

$$\frac{d^2(p_i(x))}{dx^2} = -s_i^2 \cdot z_i \cdot \varphi(z_i), \quad (4.42)$$

$$\frac{d^3(p_i(x))}{dx^3} = -s_i^3 \cdot \varphi(z_i) \cdot \big(z_i^2 - 1\big), \quad (4.43)$$

$$\frac{d\big(p_i^2(x)\big)}{dx} = -2 \cdot s_i \cdot \Phi(z_i) \cdot \varphi(z_i), \quad (4.44)$$

$$\frac{d^2\big(p_i^2(x)\big)}{dx^2} = 2 \cdot s_i^2 \cdot \varphi(z_i) \cdot [\varphi(z_i) - \Phi(z_i) \cdot z_i], \quad (4.45)$$

$$\frac{d\big(p_i^3(x)\big)}{dx} = -3 \cdot s_i \cdot \Phi^2(z_i) \cdot \varphi(z_i), \quad (4.46)$$

$$\frac{d^2\big(p_i^3(x)\big)}{dx^2} = 3 \cdot s_i^2 \cdot \Phi(z_i) \cdot \varphi(z_i) \cdot [2 \cdot \varphi(z_i) - \Phi(z_i) \cdot z_i]. \quad (4.47)$$

Finally, we just have to use (4.37)–(4.47) in order to determine the second-order adjustment formula (4.36). The resulting expression can easily be calculated with standard computer applications without the need to aggregate the terms to a single formula. Thus, we have achived our aim to derive a formula that takes the conditional skewness into account and reduces the error to $O(\sum_{i=1}^{n} w^4)$ or to $O(1/n^3)$ for homogeneous portfolios. This can best be seen for homogeneous portfolios for the special case that the gross loss rates are modeled:

$$\Delta l_2 = \frac{1}{6n^2 s^2 \varphi^2} \big[\big(x^2 - 1 + s^2 + 3xsz + 2s^2z^2\big)\big(\Phi - 3\Phi^2 + 2\Phi^3\big) \\ + s\varphi(2x + 3sz)\big(1 - 6\Phi + 6\Phi^2\big) - s^2\varphi(z - 6[\Phi z - \varphi] + 6\Phi[\Phi z - 2\varphi])\big]$$

$$
\begin{aligned}
&-\frac{1}{8n^2s^3\varphi^3}\Big[(-x-3sz)\big(\left[\Phi-\Phi^2\right][-x-sz]-s\varphi[1-2\Phi]\big)^2\\
&+2\big(\left[\Phi-\Phi^2\right][x+sz]+s\varphi[1-2\Phi]\big)\\
&\cdot\big(\left[\Phi-\Phi^2\right]\left[1-s^2\right]-s\varphi[1-2\Phi][x+sz]+s^2\varphi[z+2(\varphi-\Phi z)]\big)\Big],
\end{aligned}
\tag{4.48}
$$

with $\Phi=\Phi(z)$, $\varphi=\varphi(z)$, $z=\frac{\Phi^{-1}(PD)-\sqrt{\rho}\cdot x}{\sqrt{1-\rho}}$, $s=\frac{\sqrt{\rho}}{\sqrt{1-\rho}}$, and $x=\Phi^{-1}(1-\alpha)$.

Even if the formulas appear quite complex, both adjustments are easy to implement, fast to compute and we do not have to run Monte Carlo simulations and thereby avoid simulation noise.

4.2.2 *Numerical Analysis of the VaR-Based Granularity Adjustment*

4.2.2.1 Impact on the Portfolio-Quantile

As mentioned in Sect. 4.1, there is no concrete analysis in the literature for which type of credit portfolios the impact of portfolio name concentrations is negligible. Instead, we only essentially know that a (homogeneous) portfolio consisting of a higher number of credits incorporates less name concentration risk or that name concentrations can account for round about 13–21% additional risk if the portfolio is highly concentrated.[189] Moreover, we do not know how good the first-order or the second-order granularity adjustment formulas work for different portfolio types. Against this background, subsequently the accuracy of the ASRF formula, the first-order, and the second-order granularity adjustment will be analyzed.

At first, we discuss the general behavior of the four procedures for risk quantification of homogeneous portfolios presented in Sects. 2.5, 2.6, 2.7, 4.2.1.2, and 4.2.1.4, which are

(a) The numerically “exact” coarse grained solution (see (2.75))
(b) The fine grained ASRF solution (see (2.97))
(c) The ASRF solution with first-order adjustment (see (4.10) and (4.29))
(d) The ASRF solution with first- and second-order adjustments (see (4.30) and (4.48))

each applying the conditional probability of default (2.66) of the Vasicek model. For the comparison, we evaluate the portfolio loss distribution of a simple portfolio

[189] Cf. BCBS (2006), p. 10.

that consists of 40 credits, each with a probability of default of $PD = 1\%$ and a loss given default of $LGD = 1$. The correlation parameter is set to $\rho = 20\%$.[190] Using these parameters, we calculate the loss distribution using the "exact" solution (a) as well as the approximations (b) to (d). The results are shown in Fig. 4.1 for portfolio losses up to 30 % (12 credits) and the corresponding quantiles (of the loss distribution) starting at $\alpha = 0.7$. See Fig. 4.2 for the region of high quantiles $\alpha \geq 0.994$, which are of special interest in a VaR-framework for credit risk with high confidence levels.

It is obvious to see that the coarse grained solution (a) is not continuous since the distribution of defaults is a discrete binomial mixture whereas all other solutions (b) to (d) are "smooth" functions. This is caused by the fact that these approximations for the loss distribution assume an infinitely granular portfolio, i.e. the loss distribution is monotonous increasing and differentiable (solution (b)), or at least are derived from such an idealized portfolio ((c) and (d)).

Now, we examine the result for the VaR-figures at confidence levels 0.995 and 0.999. Using the exact, discrete solution (a), the VaR is 12.5% (or 5 credits) for the

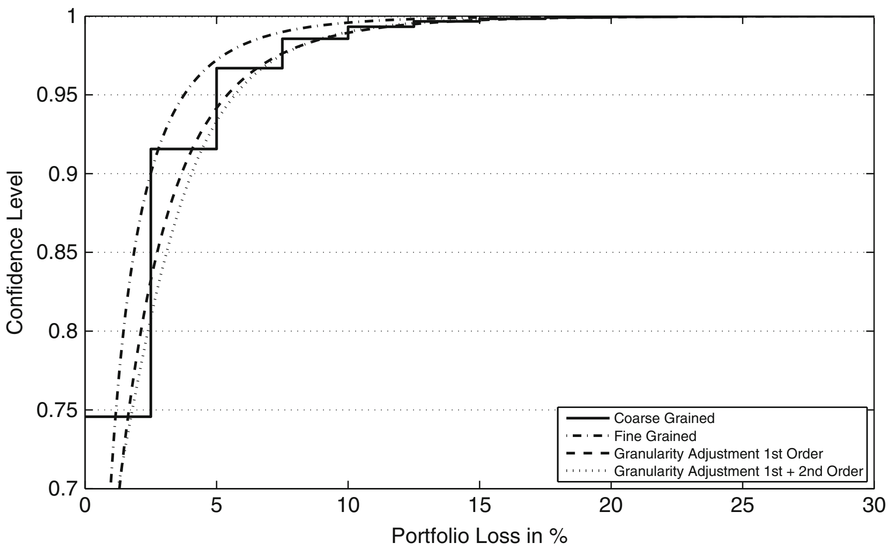

Fig. 4.1 Value at Risk for a wide range of probabilities

[190]The chosen portfolio exhibits high unsystematic risk and therefore serves as a good example in order to explain the differences of the four solutions. However, we evaluated several portfolios and basically, the results do not differ. Additionally, we claim that the general statements can also be applied to heterogeneous portfolios.

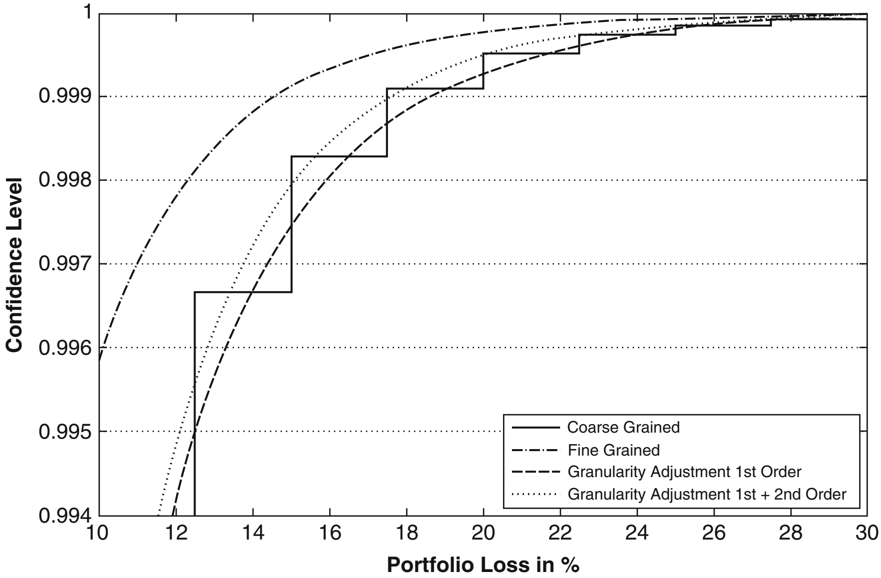

Fig. 4.2 Value at Risk for high confidence levels

0.995 quantile and 17.5% (or 7 credits) for the 0.999 quantile. Compared to this, the ASRF solution (b) exhibits significant lower losses at these confidence levels, which are 9.46% for the 0.995 quantile and 14.55% for the 0.999 quantile. Obviously, the ASRF solution underestimates the portfolio loss, since it does not take (additional) concentration risks into account. If we add the first order adjustment (c), the VaR figures increase compared to the ASRF solution (b) with values 12.55% for the 0.995 quantile and 18.59% for the 0.999 quantile. Both values are good proxies for the "true" solution (a). Especially the VaR at 0.995 confidence level is nearly exact (12.55% compared to 12.5%). However, (c) seems to be a conservative measure, since the VaR is positively biased.

Using the additional second-order adjustment (d), the VaR is lowered to 12.12% for the 0.995 quantile and 17.48% for the 0.999 quantile. In this case, the VaR at 0.999 confidence level is nearly exact (17.48% compared to 17.5%). Nonetheless, (d) is likely to be a progressive approximation for the "exact" solution (a), since the VaR is negatively biased. Summing up these first results (see also Figs. 4.1 and 4.2), using the ASRF solution (b), the portfolio distributions shift to lower losses for the VaR compared to the "exact" solution (a), since an infinitely high number of credits is presumed. Precisely, the idiosyncratic risk is diversified completely, resulting in a lower portfolio loss at high confidence levels. If the first order granularity adjustment (c) is incorporated, this effect is weakened and especially for the relevant high confidence levels the portfolio loss increases compared to the ASRF solution (b). This means that the first-order granularity adjustment is usually positive.[191] However, if the second-order granularity adjustment (d) is added, the portfolio loss distribution shifts backwards again (for high confidence levels). This can be addressed to the alternating sign of the Taylor series, as can be seen in (4.31). Since the first-order granularity adjustment is positive, the second-order adjustment tends to be negative. Thus, with incorporation of the second-order adjustment (d), the approximation of the discrete distribution of the coarse grained portfolio (a) is (in general) less conservative compared to the (only) use of the first order adjustment. However, a clear conclusion that the application of the second-order adjustment (d) in order to approximate the discrete numerical derived distribution (a) for high confidence levels outperforms the only use of the first-order adjustment (c) cannot be stated.[192]

To conclude, if we appraise the approximations for the coarse grained portfolio, we find both adjustments (c) and (d) to be a much better fit of the numerical solution in the (VaR relevant) tail region of the loss distribution than the ASRF solution, whereas the first-order adjustment is more conservative and seems to give the better overall approximation in general.

4.2.2.2 Size of Fine Grained Risk Buckets

Reconsidering the assumptions of the ASRF framework (see Sect. 2.6), we found assumption (A) – the infinite granularity assumption – to be critical in a one factor model. Thus, we investigate in detail the critical numbers of credits in homogeneous portfolios that fulfill this condition. Therefore, we have to define a critical value for the deviation of the "idealized" VaR of the ASRF solution (b) from the "true" VaR figure from solution (a) to discriminate an infinite granular portfolio from a finite granular portfolio. We do that in two ways:

Firstly, it could be argued that the fine grained approximation (2.97) in order to calculate the VaR is only adequate if its value does not exceed the "true" VaR from (2.75) of the coarse grained bucket minus a target tolerance β, both using a confidence level of 0.999. Precisely, we define a critical number $I_{c,\mathrm{per}}^{(\mathrm{ASRF})}$ of credits in the bucket, so that each portfolio with a higher number of credits than $I_{c,\mathrm{per}}^{(\mathrm{ASRF})}$ meets this specification. We use the expression[193]

---

[191] See Rau-Bredow (2005) for a counter-example for very unusual parameter values. This problem can be addressed to the use of VaR as a measure of risk which does not guarantee sub-additivity; cf. Sect. 2.2.3.

[192] By contrast, we expected a significant enhancement by using the second order adjustment like mentioned in Gordy (2004), p. 112, footnote 5.

[193] To address to the minimum number after which the target tolerance will permanently hold, we have to add the notation "for all $N \geq n$" because the function of the coarse grained VaR exhibits jumps dependent on the number of credits.

$$I_{c,\mathrm{per}}^{(\mathrm{ASRF})} = \inf\left(n : \left| \frac{VaR_{0.999}^{(\mathrm{ASRF})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right). \tag{4.49}$$

Here, we set the target tolerance β to 5%, meaning that the "true" VaR specified by coarse grained risk buckets does not differ from the analytic VaR using the fine grained solution (2.97) by more than 5% if the number of credits in the bucket reaches at least $I_{c,\mathrm{per}}^{(\mathrm{ASRF})}$.

Secondly, the fine grained approximation (b) of the VaR ("idealized" VaR) may be sufficient as long as its result using a confidence level of 0.999 does not exceed the "true" VaR as defined by solution (a) of the coarse grained bucket using a confidence level of 0.995, i.e.

$$I_{c,\mathrm{abs}}^{(\mathrm{ASRF})} = \sup\left(n : VaR_{0.999}^{(\mathrm{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(n)}\left(\tilde{L}\right)\right). \tag{4.50}$$

This definition of a critical number can be justified due to the development of the IRB-capital formula in Basel II: When the granularity adjustment (of Basel II) was cancelled, simultaneously the confidence level was increased from 0.995 to 0.999.[194] Thus, the reduction of the capital requirement by neglecting granularity was roughly compensated by an increase of the target confidence level. The risk of portfolios with a high number of credits will therefore be overestimated if we assume that the actual target confidence level is 0.995, whereas the risk for a low number of credits will be underestimated. Thus, a critical number $I_{c,\mathrm{abs}}^{(\mathrm{ASRF})}$ of credits in the bucket exists, so that in each portfolio with a higher number of credits than $I_{c,\mathrm{abs}}^{(\mathrm{ASRF})}$, the VaR can be stated to be overestimated.

The critical numbers $I_{c,\mathrm{per}}^{(\mathrm{ASRF})}$ and $I_{c,\mathrm{abs}}^{(\mathrm{ASRF})}$ for homogeneous portfolios with different parameters ρ and PD are reported in Tables 4.1 and 4.2. We do not only report the critical numbers for Basel II conditions, but also a for wide range of parameter settings that might be relevant if banks internal data are used for estimating ρ. Due to the supervisory formula, this parameter is a function of PD for corporates, sovereigns, and banks as well as for Small and Medium Enterprises (SMEs) and (other) retail exposures and remains fixed for residential mortgage exposures and revolving retail exposures.[195]

With definition (4.49), the critical numbers $I_{c,\mathrm{per}}^{(\mathrm{ASRF})}$ vary from 23 to 35,986 credits (see Table 4.1), dependent on the probability of default PD and the correlation

[194]Beside some adjustments on the correlation parameter, these were the major changes of the IRB-formula from the second to the third consultative document; see BCBS (2001a, 2003a).

[195]See Sect. 2.7 for details. In both tables, (rounded) parameters ρ due to Basel II are marked.

Table 4.1 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true VaR (see (4.49))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	35,986	23,985	5,389	5,184	4,105	3,176	2,057	1,390	988	478	370	205
3.5%	30,501	20,122	4,627	4,457	3,544	2,755	1,801	1,214	861	421	322	175
4.0%	26,051	17,272	4,054	3,851	3,076	2,402	1,563	1,077	760	375	295	161
4.5%	22,372	14,906	3,569	3,392	2,719	2,132	1,398	958	690	350	271	145
5.0%	19,669	13,160	3,153	3,047	2,412	1,928	1,273	866	628	320	255	128
5.5%	17,723	11,667	2,840	2,701	2,180	1,722	1,145	784	564	289	229	125
6.0%	15,715	10,590	2,611	2,442	1,977	1,566	1,032	711	515	264	205	116
6.5%	14,276	9,452	2,366	2,252	1,828	1,428	946	655	477	251	201	106
7.0%	12,730	8,637	2,148	2,045	1,665	1,327	869	615	457	226	185	101
7.5%	11,633	7,915	1,990	1,896	1,547	1,214	827	578	412	209	167	90
8.0%	10,657	7,272	1,813	1,761	1,414	1,133	762	527	389	206	160	87
8.5%	9,785	6,695	1,720	1,607	1,318	1,040	703	505	357	200	156	87
9.0%	9,222	6,176	1,571	1,498	1,231	992	660	460	338	183	143	80
9.5%	8,504	5,707	1,466	1,427	1,152	930	610	443	326	164	135	76
10.0%	7,853	5,281	1,399	1,334	1,079	873	597	419	304	157	132	68
10.5%	7,262	5,015	1,309	1,249	1,011	804	552	382	289	153	118	70
11.0%	6,900	4,655	1,226	1,170	949	756	532	376	285	144	120	65
11.5%	6,398	4,324	1,149	1,097	911	726	493	357	257	138	109	64
12.0%	6,099	4,127	1,103	1,053	838	684	466	332	254	135	107	58
12.5%	5,669	3,843	1,036	989	806	645	450	315	242	127	103	60
13.0%	5,419	3,677	974	952	759	622	435	299	226	117	94	53
13.5%	5,046	3,430	915	896	732	587	395	284	211	117	98	55
14.0%	4,701	3,290	882	843	706	555	391	288	201	110	87	52
14.5%	4,510	3,073	851	794	666	536	362	263	200	101	91	50
15.0%	4,331	2,954	822	767	629	519	344	250	195	108	84	51
15.5%	4,044	2,763	775	741	594	491	349	254	178	95	81	52
16.0%	3,892	2,661	731	717	589	476	324	226	186	100	78	44
16.5%	3,748	2,564	690	677	557	451	315	220	174	96	75	51
17.0%	3,507	2,403	668	639	540	427	299	225	159	86	67	42
17.5%	3,383	2,320	647	619	511	404	291	205	159	95	66	38
18.0%	3,167	2,241	611	585	496	403	277	200	152	80	70	33
18.5%	3,060	2,103	593	583	469	382	263	195	145	90	61	34
19.0%	2,959	2,034	576	551	456	362	250	186	142	85	65	35
19.5%	2,863	1,969	544	521	432	352	250	186	129	80	61	30
20.0%	2,685	1,850	529	507	420	343	244	173	133	77	57	31
20.5%	2,601	1,793	500	493	409	317	232	165	127	74	58	32
21.0%	2,522	1,739	487	466	377	326	227	170	131	73	51	26
21.5%	2,446	1,635	474	454	367	301	216	158	119	63	52	27
22.0%	2,297	1,587	448	442	368	302	211	163	123	64	53	28
22.5%	2,230	1,541	437	418	349	279	206	152	118	63	55	29
23.0%	2,167	1,498	413	408	350	280	191	145	113	57	53	30
23.5%	2,036	1,457	415	398	332	266	192	142	111	58	51	22
24.0%	1,980	1,371	393	388	324	252	193	132	98	54	49	23

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales <$ 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.2 Critical number of credits from that the exact solution at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.50))

	AAA up to AA− 0.03%	A− up to A+ 0.05%	BBB+ 0.32%	BBB 0.34%	BBB− 0.46%	BB+ 0.64%	BB 1.15%	BB− 1.97%	B+ 3.19%	B 8.99%	B− 13.01%	CCC up to C 30.85%
3.0%	5,499	3,885	997	1,019	786	678	464	329	255	165	143	123
3.5%	4,354	3,126	836	793	665	542	380	274	217	138	122	110
4.0%	3,428	2,508	701	666	564	428	308	227	184	118	103	94
4.5%	3,111	1,998	588	558	434	364	266	200	155	100	93	79
5.0%	2,436	1,830	490	466	404	308	230	175	138	92	83	70
5.5%	2,239	1,445	406	386	339	288	198	154	123	77	71	65
6.0%	1,724	1,338	380	361	283	244	170	135	109	74	69	57
6.5%	1,599	1,037	312	297	266	204	161	117	97	68	58	56
7.0%	1,489	968	294	280	220	193	138	112	85	62	57	50
7.5%	1,114	906	238	264	208	183	131	97	82	57	50	46
8.0%	1,044	681	225	214	197	152	111	93	72	52	46	42
8.5%	982	641	214	204	161	145	106	80	63	47	45	43
9.0%	925	605	203	194	153	119	102	77	61	46	39	41
9.5%	874	573	161	185	146	113	85	66	59	42	38	39
10.0%	621	543	154	147	140	109	82	64	51	38	37	38
10.5%	589	516	147	140	111	104	79	61	49	37	34	35
11.0%	559	368	141	134	107	100	76	52	48	36	31	30
11.5%	532	351	135	129	103	80	63	50	41	32	28	31
12.0%	507	335	130	124	99	77	61	49	40	32	30	28
12.5%	484	320	100	95	95	74	59	47	39	31	27	29
13.0%	463	306	96	92	91	72	57	46	38	28	29	26
13.5%	443	293	92	88	71	69	55	38	37	30	24	27
14.0%	425	281	89	85	68	67	44	37	31	27	26	24
14.5%	407	270	86	82	66	65	43	36	31	24	22	28
15.0%	261	260	83	79	64	50	42	35	30	21	23	21
15.5%	251	250	80	77	62	49	40	34	29	23	25	25
16.0%	242	241	77	74	60	47	39	33	24	23	21	22
16.5%	233	155	75	72	58	46	38	27	28	20	18	23
17.0%	224	149	55	70	56	44	37	26	23	22	22	19
17.5%	216	144	53	51	54	43	36	31	27	17	20	24
18.0%	209	139	51	49	53	42	28	25	22	19	18	20
18.5%	202	135	50	48	39	41	28	24	22	19	16	20
19.0%	195	130	48	46	37	40	27	24	18	16	16	21
19.5%	189	126	47	45	36	39	26	23	21	16	19	21
20.0%	183	122	46	44	35	38	26	23	21	18	17	17
20.5%	177	118	44	43	35	37	25	22	17	18	17	17
21.0%	172	115	43	41	34	27	24	22	20	14	15	18
21.5%	167	112	42	40	33	26	24	17	16	13	15	18
22.0%	162	108	41	39	32	26	23	21	16	15	13	19
22.5%	157	105	40	38	31	25	23	21	16	15	13	19
23.0%	153	102	39	37	30	24	22	16	15	15	13	14
23.5%	148	99	38	36	30	24	22	16	15	15	16	14
24.0%	144	97	37	36	29	23	16	16	15	13	11	15

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

factor ρ. In buckets with small probabilities of default as well as low correlation factors, the idiosyncratic risk is relatively high, so that the portfolio must be substantially bigger to meet the target. This means that in the worst case, a portfolio must consist of at least 35,986 creditors to meet the assumptions of the ASRF framework at an accuracy of 5%. The same tendency can also be found for the target tolerance specification (4.50). We get critical numbers $I_{c,\text{abs}}^{(\text{ASRF})}$ ranging from 11 to 5,499 creditors (see Table 4.2), that are substantially lower compared to the critical numbers of the target tolerance. Thus, the critical number $I_{c,\text{abs}}^{(\text{fg})}$ is less conservative. This is caused by the effect that an increase of the confidence level for VaR calculations has a high impact, especially on risk buckets with low default rates. However, since for all those obligors the ASRF assumptions (see Sect. 2.6) still have to be valid, such big risk buckets may mainly be relevant for retail exposures in practice. Furthermore, it should be mentioned that these portfolio sizes are only valid for homogeneous portfolios. For heterogeneous portfolios, these numbers can be considerably higher, especially because the exposure weights differ between the obligors and thus concentration risk will occur.[196] In order to get an impression of real-world portfolio sizes, we refer to the data of the German credit register used in Düllmann and Erdelmeier (2009). The credit register contains all bank loans exceeding €1.5 million. In September 2006, out of 1,360 reporting financial enterprises,[197] there were in total 28 german banks which had at least 1,000 registered bank loans. Even if there are also smaller loans that are not included in the data, loans for corporate, sovereigns, and banks should mostly exceed the critical size. Hence, having a look at the required number of credits in Table 4.1, most bank portfolios cannot be treated as infinitely granular. Therefore, an improvement of measuring the portfolio-VaR is indeed advisable. However, it has to be mentioned that for portfolios with debtors incorporating low credit-worthiness the ASRF solution is already sufficient for some hundred credits (or even less).

4.2.2.3 Probing First-Order Granularity Adjustment

After auditing the adequacy of the ASRF solution (b) compared to the discrete, "true" solution (a) in context of a homogeneous risk bucket, we now investigate the accuracy of the first order granularity adjustment (solution (c)). Similar to Sect. 4.2.2.2, we compare its accuracy with the discrete solution (a) but we additionally relate its result to the ASRF solution (b).

For the first (conservative) number $I_{c,\text{per}}^{(\text{1st Order Adj.})}$, we compare the analytically derived VaR including first order approximation (solution (c)) with the "true" VaR

[196] The case of heterogeneous portfolios will be analyzed in Sect. 4.2.2.5.

[197] Cf. Deutsche Bundesbank (2009).

of the discrete, binomial solution (a), both on a 0.999 confidence level. Again, we aim to meet a target tolerance of β and we get

$$I_{c,\text{per}}^{(\text{1st Order Adj.})} = \inf\left(n : \left|\frac{VaR_{0.999}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1\right| < \beta \ \forall N \in \mathbb{N}^{\geq n}\right), \text{ with } \beta = 0.05. \tag{4.51}$$

Thus, any analytically derived VaR of a risk bucket which includes more credits than $I_{c,\text{per}}^{(\text{1st Order Adj.})}$ does not differ from the “true” numerically derived VaR by more than 5%.

The results for $I_{c,\text{per}}^{(\text{1st Order Adj.})}$ for homogeneous risk buckets with a specific PD/ρ-combination are reported in Table 4.3. Obviously, the critical number varies from 7 to 6,100 credits. Compared to the ASRF solution (see Table 4.1 in Sect. 4.2.2.2), the critical values drop by 83.04% at a stretch. Precisely, we find that the number of credits that is necessary to ensure a good approximation of the “true” VaR is significantly lower with adjustment (c) than without adjustment (b). For example, a high quality retail portfolio (AAA) must consist of 5,027 compared to 26,051 credits if we neglect the first order adjustment. A medium quality corporate portfolio (BBB) must contain 106 compared to 442 credits. Thus, the minimum portfolio size should be small enough to hold for many real-world portfolios and we come to the conclusion that the first order adjustment works fine even with our conservative definition of a critical value.

Next, we relate the first order granularity adjustment (c) to the ASRF formula (b). We do that by defining a critical value $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ of credits similar to definition (4.50), but this time we proclaim that the VaR of the ASRF solution without first order granularity adjustment (b) at a confidence level of 0.999 should not exceed the VaR with first order granularity adjustment (c) at a confidence level of 0.995:

$$I_{c,\text{abs}}^{(\text{1st Order Adj.})} = \sup\left(n : VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)\right). \tag{4.52}$$

The confidence level of the ASRF solution is increased by a buffer of 4 basis points, which should incorporate the idiosyncratic risk of relatively fine-grained portfolios. If we use the first order granularity adjustment for approximating the true risk, the idiosyncratic risk of a portfolio with at $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ credits should already be included in the confidence level buffer.

The critical numbers of credits $I_{c,\text{abs}}^{(\text{1st Order Adj.})}$ are shown in Table 4.4. They contain a range from 14 to 5,170. It is interesting to note that these critical values do not differ widely from the numbers $I_{c,\text{abs}}^{(\text{fg})}$, where we compared the VaR of the ASRF solution (b) with the “true” VaR using the numerical, time-consuming discrete formula. Precisely, the average percentage difference between the critical

Table 4.3 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true VaR (see (4.51))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	6,100	4,227	879	833	693	519	337	228	152	89	63	42
3.5%	5,517	3,491	810	768	590	443	291	199	133	67	54	32
4.0%	5,027	3,192	688	653	503	413	251	174	127	60	49	28
4.5%	4,169	2,936	641	609	470	355	237	165	112	54	38	24
5.0%	3,846	2,456	546	519	401	334	205	132	107	45	37	22
5.5%	3,564	2,283	513	488	378	287	195	138	94	51	35	20
6.0%	3,317	2,129	484	460	358	272	169	121	83	46	33	20
6.5%	3,098	1,993	413	435	339	258	177	105	80	34	28	18
7.0%	2,902	1,872	392	373	322	246	154	111	77	40	29	18
7.5%	2,450	1,762	373	354	277	235	133	97	61	29	27	13
8.0%	2,309	1,494	355	338	264	203	128	84	59	35	25	16
8.5%	2,181	1,414	338	322	253	215	136	81	57	31	21	16
9.0%	2,065	1,341	323	308	242	186	118	79	55	23	23	16
9.5%	1,958	1,274	309	295	232	179	114	76	54	30	19	14
10.0%	1,861	1,212	266	253	199	172	110	74	58	22	20	14
10.5%	1,771	1,156	255	271	214	148	106	64	51	19	15	11
11.0%	1,689	1,103	245	234	206	143	92	62	44	23	15	11
11.5%	1,612	1,055	263	225	178	154	89	60	43	21	17	11
12.0%	1,541	1,010	227	217	171	133	86	52	51	18	19	11
12.5%	1,476	968	219	209	166	129	74	57	46	19	23	11
13.0%	1,414	928	211	202	160	125	81	49	40	15	12	12
13.5%	1,357	892	204	195	155	121	88	54	30	16	10	8
14.0%	1,303	858	197	188	167	117	68	41	34	17	8	8
14.5%	1,253	825	191	182	145	101	66	45	33	12	8	8
15.0%	1,206	795	185	176	141	110	64	56	28	14	15	8
15.5%	1,162	767	179	171	121	107	62	49	36	14	13	12
16.0%	1,120	740	154	166	118	104	69	37	31	16	13	9
16.5%	1,081	714	168	161	114	101	67	51	23	16	11	9
17.0%	1,044	690	145	156	125	87	58	35	30	9	11	9
17.5%	1,009	668	159	152	108	96	49	30	22	7	11	9
18.0%	976	646	154	131	105	83	55	39	18	7	9	9
18.5%	944	626	150	128	115	91	61	43	25	7	9	9
19.0%	914	606	146	124	112	79	53	28	21	13	9	9
19.5%	886	588	142	136	97	77	45	32	17	18	9	9
20.0%	859	570	123	118	95	75	44	36	20	14	9	9
20.5%	834	554	120	129	104	73	43	35	13	12	7	9
21.0%	809	538	117	112	90	63	42	30	16	10	7	9
21.5%	786	523	128	109	99	70	41	25	19	10	7	9
22.0%	764	508	111	106	86	77	51	29	22	8	7	9
22.5%	743	494	108	104	84	67	40	20	14	8	7	9
23.0%	722	481	119	114	92	57	39	36	11	8	7	9
23.5%	703	468	116	99	90	72	38	24	27	8	7	9
24.0%	684	456	101	97	88	55	32	16	18	8	7	9

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.4 Critical number of credits from that the first order adjustment at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.52))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	5,170	3,544	973	935	769	626	441	327	255	164	146	128
3.5%	4,029	2,773	774	744	615	501	356	265	209	136	122	109
4.0%	3,231	2,232	633	609	504	413	295	221	175	116	105	95
4.5%	2,650	1,836	528	508	422	347	249	188	150	101	91	85
5.0%	2,213	1,538	448	431	359	296	214	162	130	89	81	76
5.5%	1,875	1,307	385	371	310	256	186	142	114	79	72	69
6.0%	1,609	1,124	335	323	270	224	163	125	101	71	65	63
6.5%	1,395	977	295	284	238	198	145	112	91	64	60	59
7.0%	1,220	856	261	252	211	176	130	100	82	59	55	55
7.5%	1,075	757	233	225	189	158	117	91	74	54	50	51
8.0%	955	673	209	202	170	142	106	83	68	50	47	48
8.5%	853	602	189	182	154	129	96	75	62	46	44	45
9.0%	766	542	171	165	140	117	88	69	58	43	41	43
9.5%	691	490	156	151	128	108	81	64	53	40	38	41
10.0%	626	445	143	138	117	99	75	59	50	38	36	39
10.5%	570	405	131	127	108	91	69	55	46	36	34	37
11.0%	521	371	121	117	100	84	64	51	43	34	32	36
11.5%	477	340	112	108	92	78	60	48	40	32	31	34
12.0%	439	313	104	100	86	73	56	45	38	30	29	33
12.5%	404	289	96	93	80	68	52	42	36	29	28	32
13.0%	374	268	90	87	74	63	49	40	34	27	27	31
13.5%	346	248	84	81	70	59	46	37	32	26	26	30
14.0%	322	231	78	76	65	56	43	35	30	25	24	29
14.5%	299	215	74	71	61	52	41	33	29	24	24	28
15.0%	279	201	69	67	58	49	39	32	27	23	23	28
15.5%	261	188	65	63	54	47	36	30	26	22	22	27
16.0%	244	176	61	59	51	44	35	29	25	21	21	26
16.5%	229	165	58	56	48	42	33	27	24	20	20	26
17.0%	215	155	55	53	46	40	31	26	23	20	20	25
17.5%	202	146	52	50	43	38	30	25	22	19	19	25
18.0%	190	138	49	48	41	36	28	24	21	18	18	24
18.5%	180	130	46	45	39	34	27	23	20	18	18	24
19.0%	170	123	44	43	37	32	26	22	19	17	17	23
19.5%	160	116	42	41	36	31	25	21	19	17	17	23
20.0%	152	110	40	39	34	29	24	20	18	16	16	22
20.5%	144	105	38	37	32	28	23	19	17	16	16	22
21.0%	136	99	36	35	31	27	22	18	17	15	16	22
21.5%	129	94	35	34	29	26	21	18	16	15	15	22
22.0%	123	90	33	32	28	25	20	17	15	14	15	21
22.5%	117	85	32	31	27	24	19	17	15	14	15	21
23.0%	111	81	30	29	26	23	18	16	14	14	14	21
23.5%	106	78	29	28	25	22	18	15	14	13	14	21
24.0%	101	74	28	27	24	21	17	15	14	13	14	20

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

numbers of Tables 4.2 and 4.4 is less than 10%. That means that the diversification behavior of the coarse grained solution and the first order approximation is very similar, i.e. the first order adjustment is a good approximation of the idiosyncratic risk of coarse grained portfolios.

4.2.2.4 Probing Second-Order Granularity Adjustment

Finally, we want to test the approximation if the (first- and) second-order adjustment is added to the ASRF formula, leading to solution (d). Similar to Sects. 4.2.2.2 and 4.2.2.3, we firstly examine the VaR according to this new formula (d) in comparison to the "exact" VaR from the coarse grained solution (a). Additionally, we analyze its performance with respect to the ASRF solution.

Again, we calculate a critical number $I_{c,\text{per}}^{(\text{1st + 2nd Order Adj.})}$ of credits to test the approximation accuracy with reference to the coarse grained formula (a) according to the "percentaged" accuracy with a target tolerance of 5% by

$$I_{c,\text{per}}^{(\text{1st} + \text{2nd Order Adj.})} = \inf\left(n : \left| \frac{VaR_{0.999}^{(\text{1st} + \text{2nd Order Adj.})}\left(\tilde{L}\right)}{VaR_{0.999}^{(N)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right),$$
$$\text{with} \quad \beta = 0.05, \tag{4.53}$$

using the (first- and) second-order adjustment as an approximation of the coarse-grained portfolio.

The results are presented in Table 4.5. Now, the critical number of credits ranges from 17 to 10,993. Compared to the ASRF solution (a), see Table 4.1 in Sect. 4.3.4.2, the necessary number of credits to meet the requirements can be reduced by 66.5% on average. Thus, the second-order adjustment is capable to detect idiosyncratic risk caused by a finite number of debtors to a certain extent. However, if we compare the results with the ones where only the first-order adjustment is used (see Table 4.3 in Sect. 4.3.4.3), the second-order adjustment performs worse.

We are able to verify this result by analyzing the second-order adjustment (d) in comparison to the exact ASRF solution (a). Therefore we introduce a critical number $I_{c,\text{abs}}^{(\text{1.+2. Order Adj.})}$ of credits, similar to the definition (4.52) in Sect. 4.3.4.3. We get

$$I_{c,\text{abs}}^{(\text{1st +2nd Order Adj.})} = \sup\left(n : VaR_{0.999}^{(\text{ASRF})}\left(\tilde{L}\right) < VaR_{0.995}^{(\text{1st +2nd Order Adj.})}\left(\tilde{L}\right)\right). \tag{4.54}$$

Thus, for each risk bucket with at least $I_{c,\text{abs}}^{(\text{1st +2nd Order Adj.})}$ credits the idiosyncratic risk, measured by the second-order adjustment on a confidence level 0.995, is included in the confidence level premium of 4 basis points of the ASRF solution (on a confidence level 0.999).

Table 4.5 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true VaR (see (4.53))

	AAA up to AA− 0.03%	A− up to A+ 0.05%	BBB+ 0.32%	BBB 0.34%	BBB− 0.46%	BB+ 0.64%	BB 1.15%	BB− 1.97%	B+ 3.19%	B 8.99%	B− 13.01%	CCC up to C 30.85%
3.0%	10,993	7,338	1,796	1,770	1,417	1,107	746	522	392	222	185	130
3.5%	9,309	6,251	1,503	1,427	1,150	941	620	440	327	193	163	115
4.0%	7,494	5,077	1,260	1,252	1,014	802	534	384	280	167	140	103
4.5%	6,405	4,367	1,109	1,054	858	683	460	323	255	148	120	90
5.0%	5,864	3,768	979	930	761	609	414	293	225	127	115	83
5.5%	5,056	3,256	866	824	677	544	373	266	199	118	103	78
6.0%	4,362	3,021	767	730	603	486	321	242	182	107	94	70
6.5%	4,055	2,622	680	647	537	435	304	210	167	100	86	64
7.0%	3,509	2,452	641	610	478	390	260	191	147	90	76	63
7.5%	3,286	2,132	570	542	453	349	248	183	141	84	74	60
8.0%	2,844	2,006	505	481	404	332	237	158	123	79	67	55
8.5%	2,679	1,892	480	457	385	297	214	160	119	71	63	51
9.0%	2,529	1,649	457	406	343	284	193	146	109	69	57	49
9.5%	2,394	1,563	406	387	328	254	174	133	105	67	58	51
10.0%	2,077	1,484	388	370	292	243	168	128	91	60	50	42
10.5%	1,974	1,412	344	354	280	234	161	116	88	56	49	43
11.0%	1,879	1,231	330	314	269	209	145	106	81	52	48	41
11.5%	1,791	1,175	316	302	239	201	140	109	88	51	45	38
12.0%	1,710	1,123	304	290	230	194	126	99	76	52	41	39
12.5%	1,484	1,075	269	257	222	173	131	96	74	51	42	37
13.0%	1,421	1,030	259	248	214	167	127	87	63	43	43	34
13.5%	1,362	897	250	239	190	149	106	79	70	42	37	34
14.0%	1,307	861	241	230	184	144	111	76	64	39	38	31
14.5%	1,256	828	233	203	177	139	92	80	54	38	34	32
15.0%	1,208	797	206	197	172	135	97	67	61	33	35	28
15.5%	1,163	768	199	190	152	131	94	65	52	39	31	29
16.0%	1,120	741	193	184	147	127	84	74	51	34	34	30
16.5%	1,081	715	187	178	143	113	89	67	46	38	30	26
17.0%	938	690	181	173	152	120	73	56	45	33	28	26
17.5%	906	600	176	168	135	106	71	64	51	31	26	27
18.0%	876	646	155	163	131	103	69	58	43	32	24	28
18.5%	847	562	150	144	115	101	74	52	42	30	27	23
19.0%	820	544	146	140	124	98	72	51	41	26	25	23
19.5%	795	527	142	150	109	86	64	45	37	29	23	24
20.0%	770	511	138	132	106	93	57	44	33	27	26	25
20.5%	747	496	134	115	93	91	67	43	42	23	21	26
21.0%	725	482	131	125	101	80	60	39	38	21	24	26
21.5%	704	468	114	122	88	78	53	42	31	24	22	20
22.0%	684	455	124	119	96	68	57	41	34	22	22	20
22.5%	665	442	121	116	94	67	56	44	39	22	20	21
23.0%	647	430	106	101	82	73	44	32	30	20	17	22
23.5%	629	419	103	99	80	64	43	35	24	18	21	22
24.0%	613	408	101	108	78	62	43	38	29	21	18	23

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)

SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.6 Critical number of credits from that the first plus second order adjustment at confidence level 0.995 exceeds the infinite fine granularity at confidence level 0.999 (see (4.54))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	4,285	2,942	810	778	640	521	367	272	214	140	125	114
3.5%	3,266	2,254	633	609	503	411	292	218	173	115	104	97
4.0%	2,560	1,776	508	489	406	333	238	180	143	97	89	84
4.5%	2,050	1,429	417	401	334	275	198	151	121	83	77	75
5.0%	1,671	1,170	347	335	279	231	168	128	103	73	68	67
5.5%	1,380	971	294	283	237	196	144	111	90	64	60	61
6.0%	1,153	815	251	242	203	169	124	96	79	57	54	56
6.5%	973	691	216	209	176	147	109	85	70	52	49	51
7.0%	827	590	188	182	153	128	96	75	62	47	44	48
7.5%	708	507	164	159	135	113	85	67	56	43	41	44
8.0%	610	439	145	140	119	100	76	60	50	39	38	42
8.5%	527	382	128	124	106	89	68	54	46	36	35	39
9.0%	458	333	114	110	94	80	61	49	42	33	32	37
9.5%	399	292	102	98	84	72	55	45	38	31	30	35
10.0%	349	257	91	88	76	65	50	41	35	29	28	33
10.5%	306	226	82	79	68	59	46	37	32	27	27	32
11.0%	268	200	74	72	62	53	42	34	30	25	25	31
11.5%	264	177	67	65	56	48	38	32	28	24	24	29
12.0%	271	156	60	59	51	44	35	29	26	22	22	28
12.5%	266	173	55	53	46	40	32	27	24	21	21	27
13.0%	257	172	50	48	42	37	30	25	22	20	20	26
13.5%	248	167	45	44	39	34	27	23	21	19	19	25
14.0%	238	162	41	40	36	31	25	22	20	18	18	24
14.5%	229	156	38	37	33	29	24	20	18	17	18	24
15.0%	219	150	34	34	30	26	22	19	17	16	17	23
15.5%	210	144	38	36	27	24	20	18	16	15	16	22
16.0%	201	139	38	36	28	23	19	17	15	15	15	22
16.5%	193	133	37	36	29	21	18	16	14	14	15	21
17.0%	185	128	37	35	29	22	16	15	14	13	14	21
17.5%	177	123	36	34	28	23	15	14	13	13	14	20
18.0%	170	118	35	33	28	23	14	13	12	12	13	20
18.5%	163	113	34	33	27	22	13	12	12	12	13	19
19.0%	156	109	33	32	26	22	15	11	11	11	12	19
19.5%	150	105	32	31	26	21	15	11	10	11	12	19
20.0%	145	101	31	30	25	21	15	10	10	11	12	18
20.5%	139	97	30	29	24	20	15	10	9	10	11	18
21.0%	134	94	29	28	24	20	14	9	9	10	11	18
21.5%	129	90	28	27	23	19	14	10	8	10	11	17
22.0%	124	87	27	26	22	19	14	10	8	9	10	17
22.5%	120	84	26	26	22	18	14	10	8	9	10	17
23.0%	115	81	26	25	21	18	13	10	7	9	10	16
23.5%	111	78	25	24	20	17	13	10	7	8	9	16
24.0%	108	75	24	23	20	17	13	10	7	8	9	16

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

The critical numbers presented in Table 4.6 range from 7 to 4,285. Obviously, these results are considerably higher than those of Table 4.4 and therefore the predefined target value of accuracy is reached with lower numbers of credits. Thus, the idiosyncratic risk is underestimated with the second order adjustment compared to the numerical "true" solution (a) (see the results in Sect. 4.2.2.2) and is not measured with such a high accuracy as the first order adjustment does (see Sect. 4.2.2.3). Concretely, this value is reduced by averaged 32.7% credits.

To conclude, the second-order adjustment (d) converges faster to the asymptotic value of the ASRF solution (b), which confirms the findings of Sect. 4.2.2.1. A possible reason is that the VaR measure using the first order approximation may be "corrected" into the direction of the ASRF solution by incorporating the second order adjustment. The possibility of this behavior is given due to the alternating sign in the derivatives of VaR; see (4.31).[198] Thus, taking more derivatives into account could solve the problem but would lead to even more uncomfortable equations.[199] Despite these theoretical questions, it can be stated that in homogeneous portfolios, an excellent approximation of the true VaR can be achieved with the granularity adjustment.

4.2.2.5 Probing Granularity for Inhomogeneous Portfolios

The previous analyses showed that the granularity adjustment works fine for homogeneous portfolios. In this section, we test if the approximation accuracy of the presented general formulas will hold for portfolios consisting of loans with different exposures and credit qualities. This means that the credits in the portfolio vary in exposure weight and in probability of default, and we analyze if the portfolio loss for coarse grained portfolios could still be quantified satisfactorily by the granularity adjustment.

Concretely, we examine high quality portfolios with probabilities of default ranging from 0.02 to 0.79% and lower quality portfolios with probabilities of default ranging from 0.2 to 7.9%. Additionally, we define a basic risk bucket consisting of 20 loans with exposures between €35 and 200 million.[200] In order to measure the portfolio size with respect to concentration risk, we use the effective number of loans n^* (see (2.87)), rather than the number of loans n. Consequently, this effective number is more than 25% below the true number of credits.

[198]This is true not only for the first five derivatives but also for all following derivatives; see the general formula for all derivatives of VaR in (4.213).

[199]However, we also have to take into consideration that the Taylor series is potentially not convergent at all or does not converge to the correct value. For a further discussion see Martin and Wilde (2002) and Wilde (2003).

[200]The used portfolio is based on Overbeck (2000), see also Overbeck and Stahl (2003), but reduced to 20 loans to achieve more test portfolios with a small number of credits.

A variation of portfolio size is reached by reproducing the loans of the basic risk bucket so that portfolios with 40, 60, ..., 400, 800, 1,600 and 4,000 loans result. Using an asset correlation $\rho = 20\%$ and a confidence level of 0.999, we compute the granularity add-on with the presented first-order and second-order adjustment. Because the exact value cannot be determined analytically for heterogeneous portfolios, we compute the "true" VaR with Monte Carlo simulations using three million trials.[201] Finally, we compare this "true" VaR with the ASRF solution, so that we receive the granularity add-on.

The simulated results for the granularity add-on for high and low quality portfolios are presented in Fig. 4.3 (see the circles and dots). Therefore, the add-on for the minimum size of 40 loans with $1/n^* \approx 0.035$ is 5.0% (6.2%) for the high (low) quality portfolio. This is equal to a relative correction of +112.5% (+30.5%) compared to a hypothetical infinitely fine grained portfolio. This shows again the relatively high impact of idiosyncratic risk in small high quality portfolios. With shifting to bigger sized portfolios, the effective number of credits shifts to zero and

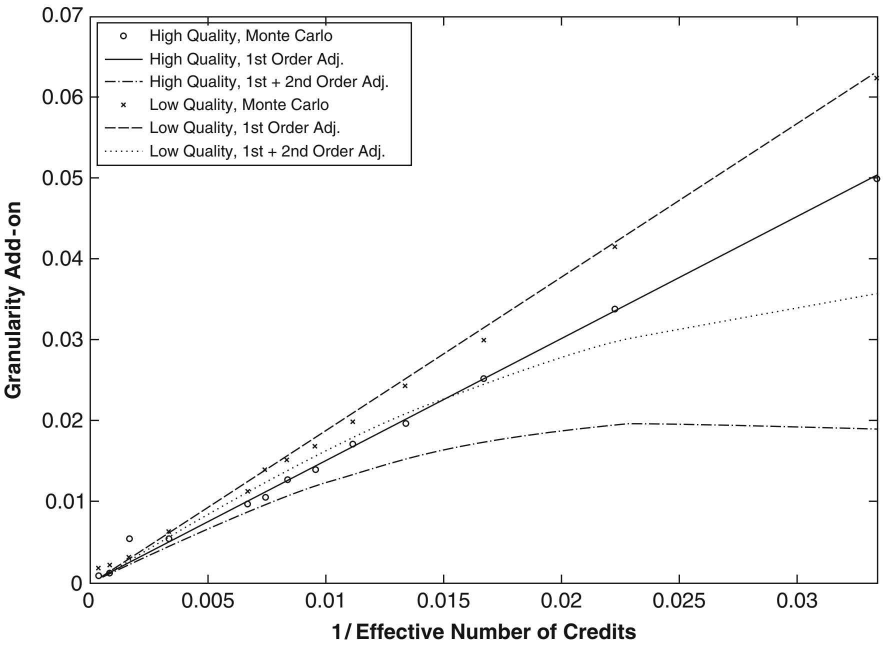

Fig. 4.3 Granularity add-on for heterogeneous portfolios calculated analytically with first-order (*solid lines*) and second-order (*dotted lines*) adjustments as well as with Monte Carlo simulations (+ and o) using three million trials

[201]Due to the high number of trials, which corresponds to 3,000 hits in the tail for a confidence level of 0.999, the simulation noise should be negligible.

the granularity add-on decreases almost exactly linear in terms of $1/n^*$ – even for high quality portfolios. This result is contrary to Gordy (2003), who exhibits a concave characteristic of the granularity add-on. This might be due to the fact that Gordy (2003) uses a CreditRisk$^+$ framework, whereas we analyze the effect of the granularity with the CreditMetrics one-factor model that is consistent with the Basel II assumptions. Summing up, the granularity add-on in Fig. 4.3 can be approximated with a linear function. Indeed, the (linear) first order adjustment is a very good approximation for heterogeneous portfolios of high as well as low quality. Just like in the previous sections, the second-order adjustment leads to a reduction of the granularity add-on. Thus, it can be characterized as less conservative, but comparing the results we strongly recommend the first-order adjustment.

4.3 Measurement of Name Concentration Using the Risk Measure Expected Shortfall

4.3.1 Adjusting for Coherency by Parameterization of the Confidence Level

As shown in Sect. 2.2.3, the commonly used VaR is not coherent because it is not necessarily subadditive. As long as we stay in the ASRF framework, this characteristic is not problematic because in this context, the VaR is exactly additive.[202] However, if we leave the ASRF framework, this behavior is not guaranteed anymore.[203] Nevertheless, many contributions that deal with concentration risk in the context of Basel II use the VaR to quantify credit risk without questioning the risk measure (possibly to be consistent with the ASRF framework), even if the subadditivity could get problematic if concentration risk is considered.[204] Thus, it could be beneficial to change the measure of risk, e.g. to use the coherent Expected Shortfall (ES). However, we cannot simply replace the VaR with the ES since the resulting difference in the capital requirements would not only stem from a more convenient measurement of concentration risk but also from the fact that the ES exceeds the VaR by definition. Against this background, we propose a procedure how the ES can be used instead of the VaR for the measurement of credit risk by accurately choosing a different confidence level. Based on this result, we analyze the performance of the ASRF formula, the first-order, and the second-order granularity adjustment when the ES is used instead of the VaR in Sect. 4.3.4 after deriving both adjustment formulas in Sect. 4.3.2.

[202] Cf. Sect. 2.6.

[203] This is true for a violation of both the granularity and the single risk factor assumption.

[204] See e.g. Heitfield et al. (2006), Cespedes et al. (2006), Düllmann (2006), as well as Düllmann and Masschelein (2007).

Before we change the risk measure, we will study the characteristics of the VaR for credit portfolios and analyze the need for using the ES. For our analyses, we continue to omit the first assumption of the ASRF framework leading to a finite granularity and calculate the VaR as well as the ES within the binomial model of Vasicek and the ASRF framework.

We start with computing the VaR at a confidence level $\alpha = 0.999$ for non-asymptotic portfolios with $PD = 0.5\%$ and $\rho = 20\%$. In Fig. 4.4, the VaR for the ASRF framework and for the Vasicek binomial model is plotted in the cases of $n = 1$ to $n = 300$ homogeneous credits. The VaR for an infinite number of credits is 9.1%. For a finite number of credits, the risk is higher because the unsystematic risk cannot be diversified. The problem is that the risk should be monotonously decreasing with a higher number of credits ("monotonicity of specific risk-property"[205]) but this behavior is not reflected by the VaR as a risk measure. Instead, we find that the VaR follows a downward sloping "saw-toothed" pattern. Although the sub-additivity axiom is not violated in the example, it is obvious that the measured risk should not increase with a higher number of credits and thus a better diversification. It is also possible to construct superadditive examples with a different parameter setting but this example gives a clear demonstration that it is problematic to use the VaR if there is concentration risk such as name concentration.

The saw-toothed pattern can also be explained intuitively: In the 99.9% worst-case scenario one credit out of 1, 2, 3, 4, or 5 credits defaults, which leads to a VaR of 1, 1/2, 1/3, 1/4, or 1/5. If the size of the portfolio is increased further, one additional credit defaults in the 99.9% scenario. Thus, the VaR increases from $1/5 = 20\%$ to $2/6 = 33.\bar{3}\%$. If additional credits are added to the portfolio, the

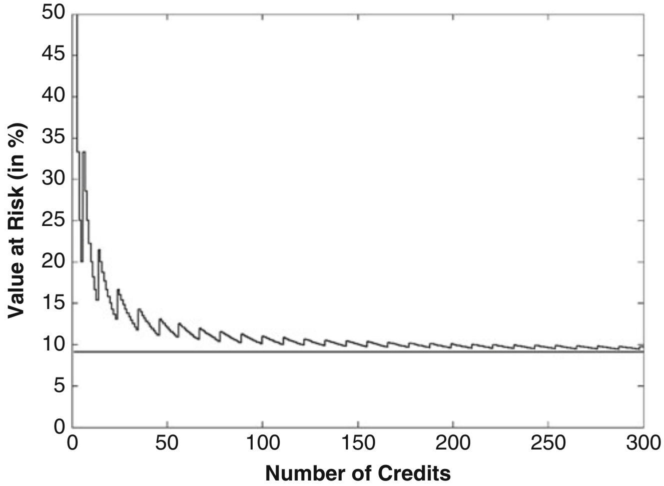

Fig. 4.4 Value at Risk in the ASRF and the Vasicek model

[205]See Albanese and Lawi (2004), p. 215, for this property of a reasonable risk measure.

VaR will increase until a third credit defaults in the considered 99.9% scenario, and so on. From a probabilistic perspective, the demonstrated problems are mainly a result of the deviation for discrete distributions $\mathbb{P}[\tilde{L} \leq VaR_\alpha(\tilde{L})] - \alpha > 0$, which is mostly decreasing with additional credits but jumps to a higher value when the difference would (theoretically) go below zero.[206] Against this background, it could be tried to define the VaR differently from the common definition of the (lower) VaR (2.12). Also the upper VaR definition (2.13) does not solve the problem. However, if the VaR was defined as the maximal loss in the best $100 \cdot \alpha\%$ scenarios

$$VaR_\alpha^{(-)}\left(\tilde{L}\right) = \sup\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \leq l\right] < \alpha\right\} \tag{4.55}$$

instead of the minimal loss in the worst $100 \cdot (1 - \alpha)\%$, we have the contrary case of a negative deviation $\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right] - \alpha < 0$. If we rewrite the common VaR definition as

$$VaR_\alpha^{(+)}\left(\tilde{L}\right) = \inf\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \leq l\right] \geq \alpha\right\} = \sup\left\{l \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l\right] < \alpha\right\}, \tag{4.56}$$

it is obvious to see that the VaR from definition (4.55) is always below the VaR from definition (4.56). In the considered case of n homogeneous credits the difference between both definitions always equals[207]

$$VaR_\alpha^{(+)} - VaR_\alpha^{(-)} = \frac{1}{n}. \tag{4.57}$$

As the positive deviation $p^{(+)} := \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \alpha > 0$ is high when the negative deviation $p^{(-)} := \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right] - \alpha < 0$ is small, we could define an *interpolated Value at Risk* $VaR^{(\mathrm{int})}$ as follows:

$$\begin{aligned} VaR_\alpha^{(\mathrm{int})} &= \frac{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \alpha}{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right]} VaR_\alpha^{(-)} \\ &\quad + \frac{\alpha - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right]}{\mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(+)}\right] - \mathbb{P}\left[\tilde{L} \leq VaR_\alpha^{(-)}\right]} VaR_\alpha^{(+)} \\ &= \frac{p^{(+)}}{p^{(+)} - p^{(-)}} VaR_\alpha^{(-)} - \frac{p^{(-)}}{p^{(+)} - p^{(-)}} VaR_\alpha^{(+)}. \end{aligned} \tag{4.58}$$

[206] Of course the definition of the VaR does not allow a negative deviation and the VaR jumps to a higher value instead.

[207] See Appendix 4.5.11.

In Fig. 4.5, this interpolated VaR as well as $VaR_{\alpha}^{(+)}$, $VaR_{\alpha}^{(-)}$ and the ASRF solution are plotted. We find that the saw-toothed pattern, which is contradictory to the "monotonicity of specific risk-property", almost vanishes for the interpolated VaR, especially if we do not consider a very small number of credits. Thus, against the background of name concentration risk, definition (4.58) seems to be much less problematic than the common VaR definition (4.56).

For comparison, we also compute the ES for the identical portfolio setting. For calculation of the ES within the Vasicek model, we have to apply (2.76). The ES in the Basel II framework can be calculated with[208]

$$ES_{\alpha}^{(\mathrm{Basel})}\left(\tilde{L}\right) = \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\right), \tag{4.59}$$

which is based on the identity (2.93) of the ES within the ASRF framework and the conditional PD of the Vasicek model. Thus, (4.59) relies on the same assumptions as the Basel II formula (2.97) but uses the ES instead of the VaR for measuring the risk. As illustrated in Fig. 4.6, the ES satisfies the "monotonicity of specific

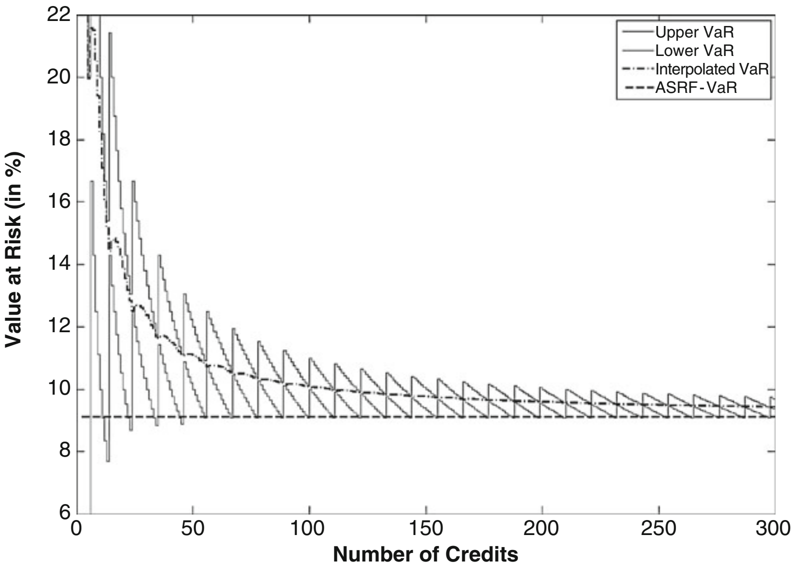

Fig. 4.5 Different Value at Risk measures in the Vasicek model

[208]See Appendix 4.5.12.

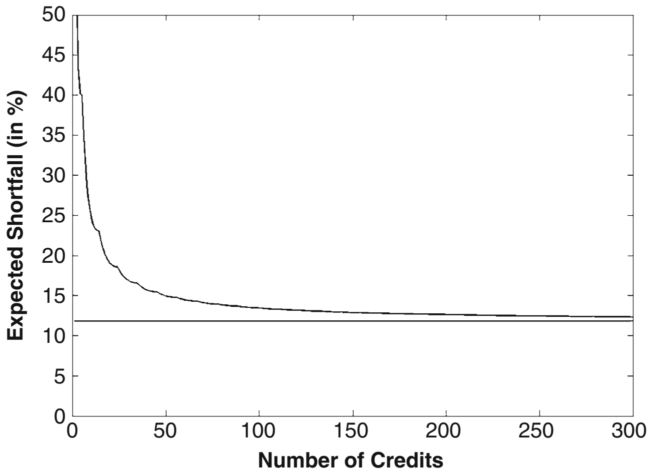

Fig. 4.6 Expected Shortfall in the ASRF and the Vasicek model

risk-property". This is one relevant advantage compared to the VaR, even if the VaR definition (4.58) is applied. Although this new VaR definition is already an improvement compared to the common definition, there are still some (minor) violations of the "monotonicity of specific risk-property", and the lack of subadditivity is still existent. Against this background, it could be beneficial to change the risk measure from VaR to ES if the portfolio contains concentration risk.[209] However, the measured economic capital would be significantly higher if it is determined on the basis of the ES instead of the VaR (by the use of the same confidence level), what is not the intended consequence of the change of the risk measure. In our example even the ASRF solution rises from 9.1% to 11.81%. Instead, we would only like to use the appreciated properties for concentration risk without being bound to increase the amount of economic capital. Therefore, the confidence level will be adjusted as described subsequently.

If we change the risk measure, we have to ensure that the new risk measure (the ES), on the one hand, is consistent with the framework presented in Pillar 2 of Basel II to get meaningful results for additional capital requirements stemming from concentration risk. On the other hand, the new risk measure should still match the capital requirements of Pillar 1 if the portfolio under consideration fulfills the assumptions of the ASRF framework; i.e. in the context of the ASRF framework, the capital requirements should not differ, regardless of whether the risk is measured by the VaR or by the ES. Therefore, we examine the VaR at the given

[209] As mentioned in Sect. 2.6, the VaR is exactly additive and therefore unproblematic in the context of the ASRF framework.

Table 4.7 Confidence level for the ES so that the ES is matched with the VaR with confidence level 0.999 for portfolios of different quality

Portfolio type/quality	$VaR_{0.999}$ and ES_{α} (%)	Confidence level α (ES) (%)
(I) AAA only	0.57	99.672
(II) Very high	6.12	99.709
(III) High	7.59	99.711
(IV) Average	12.94	99.719
(V) Low	20.89	99.726
(VI) Very low	23.30	99.727
(VII) CCC only	57.00	99.741

confidence level 0.999 for several (infinitely granular) bank portfolios of different quality. As a next step, we determine the confidence level of the ES that is necessary to match the results for both risk measures. We define this ES-confidence level α $(= \alpha(ES))$ implicitly as

$$ES_{\alpha}^{(\text{Basel})}\left(\tilde{L}\right) = VaR_{0.999}^{(\text{Basel})}\left(\tilde{L}\right), \tag{4.60}$$

with $VaR_{0.999}^{(\text{Basel})}$ given by (2.97) and $ES_{\alpha}^{(\text{Basel})}$ presented in (4.59).

Firstly, we investigate the extreme cases that all creditors of a bank have a rating of (I) AAA or (VII) CCC.[210] As can be seen in Table 4.7, the ES-confidence level must be in a range between 99.67% and 99.74%. Using these confidence levels, the economic capital is almost identical, regardless of whether the VaR or the ES is used.

Additionally, we use five portfolios with different credit quality distributions (very high, high, average, low, and very low) that are visualized in Fig. 4.7.[211] All resulting confidence levels are between 99.71% and 99.73% with mean 99.72%. Even if there is some interconnection between the confidence level and the portfolio quality, an ES-confidence level of $\alpha = 99.72\%$ seems to be accurate for most real-world portfolios.

4.3.2 *Considering Name Concentration with the Granularity Adjustment*

4.3.2.1 First-Order Granularity Adjustment for One-Factor Models

As argued in Sect. 4.3.1, the VaR can be a problematic risk measure if the assumptions of the ASRF framework, which includes the infinite granularity assumption (A)

[210]We use the idealized default rates from Standard and Poors, see Brand and Bahar (2001), ranging from 0.01% to 18.27%, but the results do not differ widely for different values.

[211]The portfolios with high, average, low, and very low quality are taken from Gordy (2000). We added a portfolio with very high quality.

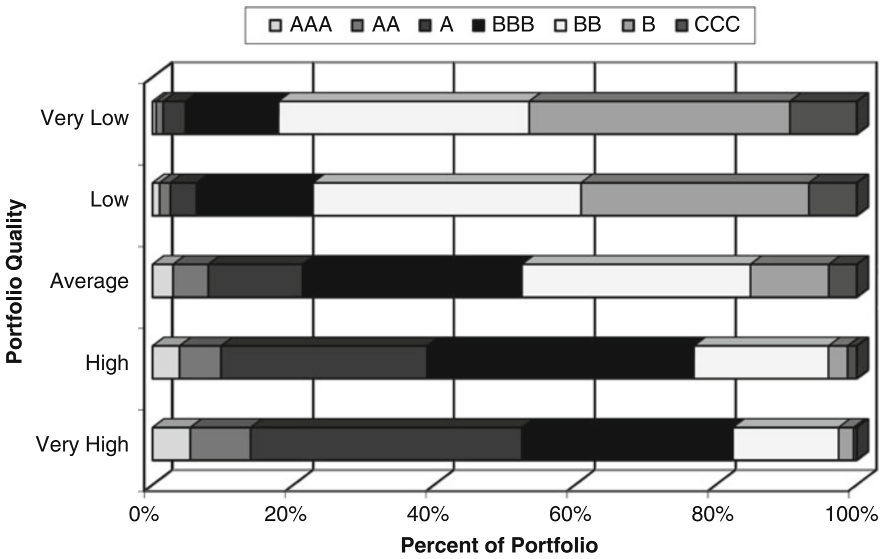

Fig. 4.7 Portfolio quality distributions

of Sect. 2.6, are not fulfilled anymore. Based on the methodology of Sect. 4.3.1, we know which confidence level is adequate if credit risk and especially concentration risk is measured on the basis of the more convenient ES instead of the VaR. However, the approximation formulas of Sect. 4.2.1 are only valid for the VaR. Thus, the ES-based granularity adjustment formulas will be derived subsequently. While the first-order granularity adjustment is already known in the literature, the second-order adjustment is a new result. The principle behind the granularity adjustment remains unchanged, regardless of whether the VaR or the ES is used as the risk measure. Thus, using the abbreviation

$$\tilde{L} = \mathbb{E}\big(\tilde{L}\,|\,\tilde{x}\big) + \big[\tilde{L} - \mathbb{E}\big(\tilde{L}\,|\,\tilde{x}\big)\big] =: \tilde{Y} + \lambda\tilde{Z}, \tag{4.61}$$

we perform a Taylor-series expansion around the systematic loss at $\lambda = 0$, leading to

$$\begin{aligned} ES_\alpha\big(\tilde{L}\big) &= ES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big) \\ &= ES_\alpha\big(\tilde{Y}\big) + \lambda\left[\frac{dES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2!}\left[\frac{d^2ES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda^2}\right]_{\lambda=0} \\ &\quad + \cdots + \frac{\lambda^m}{m!}\left[\frac{d^mES_\alpha\big(\tilde{Y} + \lambda\tilde{Z}\big)}{d\lambda^m}\right]_{\lambda=0} + \cdots. \end{aligned} \tag{4.62}$$

According to Sect. 4.2.1.1, the first-order adjustment can be calculated as the Taylor series expansion up to the quadratic term. With respect to Wilde (2003) and Rau-Bredow (2004), the needed first and second derivative of ES are given as[212]

$$\left.\frac{dES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda}\right|_{\lambda=0} = \mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y} > q_\alpha\left(\tilde{Y}\right)\right], \tag{4.63}$$

$$\left.\frac{d^2ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda}\right|_{\lambda=0} = \frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y} = q_\alpha\left(\tilde{Y}\right)\right]}{1-\alpha}. \tag{4.64}$$

Similar to the VaR, the first derivative is zero:

$$\begin{aligned}\mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y} > q_\alpha\left(\tilde{Y}\right)\right] &= \frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}-\mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)|\tilde{Y} > q_\alpha\left(\tilde{Y}\right)\right]\\ &= \frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}\,|\,\tilde{Y} > q_\alpha\left(\tilde{Y}\right)\right] - \frac{1}{\lambda}\cdot\mathbb{E}\left[\tilde{L}\,|\,\tilde{Y} > q_\alpha\left(\tilde{Y}\right)\right] = 0.\end{aligned} \tag{4.65}$$

With

$$\begin{aligned}&\tilde{Y} = q_\alpha\left(\tilde{Y}\right)\\ &\Leftrightarrow \tilde{x} = q_{1-\alpha}(\tilde{x})\end{aligned} \tag{4.66}$$

and

$$\lambda^2\cdot\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\lambda\tilde{Z}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}-\tilde{Y}\,|\,\tilde{Y}\right] = \mathbb{V}\left[\tilde{L}\,|\,\tilde{Y}\right], \tag{4.67}$$

the quadratic term of the Taylor series expansion (4.62) is equivalent to

$$\begin{aligned}\Delta l_1 &= \frac{\lambda^2}{2}\left(\frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\,\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y} = q_\alpha\left(\tilde{Y}\right)\right]}{1-\alpha}\right)\\ &= -\frac{1}{2}\frac{f_Y\left(q_\alpha\left(\tilde{Y}\right)\right)\,\mathbb{V}\left[\tilde{L}\,|\,\tilde{x} = q_{1-\alpha}(\tilde{x})\right]}{1-\alpha}.\end{aligned} \tag{4.68}$$

Using[213]

$$f_Y(y) = -f_x(x)\frac{1}{dy/dx}, \tag{4.69}$$

[212]The derivatives of ES are derived in Appendix 4.5.13 and 4.5.14.

[213]Cf. (4.8).

the first-order granularity adjustment results in

$$ES_\alpha^{(n)} \approx ES_\alpha^{(\text{ASRF})} + \Delta l_1 =: ES_\alpha^{(\text{1st Order Adj.})}$$
$$\text{with } \Delta l_1 = -\frac{1}{2(1-\alpha)} \frac{f_x(x)\, \mathbb{V}\left[\tilde{L} \,|\, \tilde{x} = q_{1-\alpha}(\tilde{x})\right]}{\frac{d}{dx}\mathbb{E}\left[\tilde{L} \,|\, \tilde{x} = x\right]\big|_{x=q_{1-\alpha}(\tilde{x})}}. \tag{4.70}$$

Analogous to the VaR-based first-order adjustment, the ES-based term Δl_1 is linear in terms of $1/n$, which means that the measured idiosyncratic risk component is halved if the number of credits is doubled. Furthermore, the adjustment formula takes the conditional variance into consideration but neglects all higher conditional moments. Thus, incorporating the add-on formula (4.70) leads to a reduction of the error from $O(1/n)$ to $O(1/n^2)$.

4.3.2.2 First-Order Granularity Adjustment for the Vasicek Model

It is straightforward to calculate the ES-based granularity adjustment for the Vasicek model. This means that the conditional PD is assumed to be given by

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.71}$$

and the systematic factor is standard normally distributed, which is analogous to Sect. 4.2.1.2. If we want to calculate the granularity adjustment (4.70), we can use the expression for the conditional variance and the derivative of the conditional expectation $d\mu_{1,c}/dx$ from Sect. 4.2.1.2. This directly leads to the formula for the ES-based granularity adjustment within the Vasicek model:

$$\begin{aligned}\Delta l_1 &= -\frac{1}{2(1-\alpha)} \frac{\varphi \eta_{2,c}}{d\mu_{1,c}/dx}\bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &= \frac{\varphi\left(\Phi^{-1}(1-\alpha)\right)}{2(1-\alpha)} \frac{\sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right)\cdot \Phi(z_i) - ELGD_i^2 \cdot \Phi^2(z_i)\right]}{\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{\sqrt{\rho_i}}{\sqrt{1-\rho_i}} \cdot \varphi(z_i)},\end{aligned} \tag{4.72}$$

with $z_i = \frac{\Phi^{-1}(PD_i)+\sqrt{\rho_i}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_i}}$, which can be simplified for homogeneous portfolios to

$$\Delta l_1 = \frac{1}{2n} \frac{\varphi\left(\Phi^{-1}(1-\alpha)\right)}{(1-\alpha)} \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \frac{\Phi(z)}{\varphi(z)} \left(\frac{ELGD^2 + VLGD}{ELGD} - ELGD\cdot\Phi(z)\right), \tag{4.73}$$

with $z = \frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}$.

4.3.2.3 Second-Order Granularity Adjustment for One-Factor Models

In order to reduce the approximation error for portfolios consisting of a small number of credits, additional elements of the Taylor-series expansion (4.62) will be calculated and analyzed subsequently. Thus, we derive all terms of order $O(1/n^2)$, which is analogous to Sect. 4.3.2.3 for the VaR-based granularity adjustment. As a consequence, not only the conditional variance but also the conditional skewness is taken into account. The resulting expression for the ASRF solution including the second-order granularity adjustment Δl_2 is

$$VaR_\alpha^{(\text{1st + 2nd Order Adj.})} = VaR_\alpha^{(\text{ASRF})} + \Delta l_1 + \Delta l_2, \tag{4.74}$$

where Δl_2 represents the $O(1/n^2)$ elements of (4.62). We already know from Appendix 4.5.8 that the third and a part of the fourth element of the Taylor series are the relevant terms for the second-order adjustment.[214] As can immediately be seen from the Taylor series expansion (4.62), the third and the fourth derivatives of ES are required for the calculation of the additional terms. Based on the formula for all derivatives of VaR, it is possible to determine a formula for arbitrary derivatives of ES. This general formula is derived in Appendix 4.5.13,[215] but for our purposes it is sufficient to use a formula for the first five derivatives of ES:[216]

$$\begin{aligned}\left.\frac{d^m ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m \cdot \frac{1}{1-\alpha} \cdot \Bigg(\frac{d^{m-2}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right) f_Y(y)\right)}{dy^{m-2}} \\ &\quad \left. - \kappa(m) \cdot \left[\frac{1}{f_Y(y)} \cdot \frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right) f_Y(y)\right)}{dy} \cdot \frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right) f_Y(y)\right)}{dy^{m-3}}\right]\Bigg)\right|_{y=q_\alpha(\tilde{Y})},\end{aligned} \tag{4.75}$$

with $\kappa(1) = \kappa(2) = 0,\ \kappa(3) = 1,\ \kappa(4) = 3$, and $\kappa(5) = 10$.

With these derivatives and due to

$$\lambda^m \cdot \mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)\big|_{y=q_\alpha(\tilde{Y})} = \eta_m\left[\tilde{L}\,|\,\tilde{Y}=y\right]\big|_{y=q_\alpha(\tilde{Y})} =: \eta_m(y)\big|_{y=q_\alpha(\tilde{Y})}, \tag{4.76}$$

[214] The explanations regarding the order of the derivatives of VaR in Appendix 4.5.8 are valid for the derivatives of ES, too.

[215] See also Wilde (2003).

[216] See Appendix 4.5.14.

the second-order adjustment for one-factor models is given as

$$\begin{aligned}\Delta l_2 &= \frac{(-1)^3}{3!}\frac{1}{1-\alpha}\left[\frac{d(\eta_3(y)f_Y(y))}{dy}\right] \\ &\quad + \frac{(-1)^4}{4!}\frac{1}{1-\alpha}\left[-3\left(\frac{1}{f_Y(y)}\cdot\frac{d(\eta_2(y)f_Y(y))}{dy}\cdot\frac{d(\eta_2(y)f_Y(y))}{dy}\right)\right]\Bigg|_{y=q_\alpha(\tilde{Y})} \\ &= -\frac{1}{6(1-\alpha)}\left[\frac{d}{dy}(\eta_3(y)f_Y(y))\right] - \frac{1}{8(1-\alpha)}\frac{1}{f_Y(y)}\left[\frac{d}{dy}(\eta_2(y)f_Y(y))\right]^2\Bigg|_{y=q_\alpha(\tilde{Y})}.\end{aligned} \tag{4.77}$$

Using $f_Y = -\frac{f_x}{dy/dx}$ and recalling that $\eta_m(y)|_{y=q_\alpha(\tilde{Y})} = \eta_m(\tilde{L}\,|\,\tilde{x}=x)\big|_{x=q_{1-\alpha}(\tilde{x})} =: \eta_{m,c}\big|_{x=q_{1-\alpha}(\tilde{x})}$ (cf. (4.9)), this leads to

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)}\frac{1}{dy/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}f_x}{dy/dx}\right) \\ &\quad + \frac{1}{8(1-\alpha)}\frac{dy/dx}{f_x}\left[\frac{1}{dy/dx}\frac{d}{dx}\left(\frac{\eta_{2,c}f_x}{dy/dx}\right)\right]^2\Bigg|_{x=q_{1-\alpha}(\tilde{x})} \\ &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}f_x}{d\mu_{1,c}/dx}\right) \\ &\quad + \frac{1}{8(1-\alpha)}\frac{1}{f_x}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}f_x}{d\mu_{1,c}/dx}\right)\right]^2\Bigg|_{x=q_{1-\alpha}(\tilde{x})},\end{aligned} \tag{4.78}$$

which is our result for the ES-based second-order granularity adjustment in general form. As mentioned before, this adjustment formula is of order $O(1/n^2)$ because both the conditional skewness and the squared conditional variance are of this order.

4.3.2.4 Second-Order Granularity Adjustment for the Vasicek Model

As in Sect. 4.3.2.2 for the first-order adjustment, we now specify the second-order adjustment for the Vasicek model. Thus, we use the conditional PD of the Vasicek model

$$p_i(x) = \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right) \tag{4.79}$$

and assume that the systematic factor is normally distributed. Due to the latter assumption, the second-order granularity adjustment (4.78) can be expressed as

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)} \frac{1}{d\mu_{1,c}/dx} \frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right) \\ &\quad + \frac{1}{8(1-\alpha)} \frac{1}{\varphi} \frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2 \Bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &=: \Delta l_{2,1} + \Delta l_{2,2}\big|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.80}$$

As presented in Appendix 4.5.15, this leads to a second-order adjustment of

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)} \frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2}\left[\frac{d\eta_{3,c}}{dx} - \eta_{3,c}\left(x - \frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right] \\ &\quad + \frac{1}{8(1-\alpha)} \frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\left[\frac{d\eta_{2,c}}{dx} - \eta_{2,c}\left(x - \frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2 \Bigg|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.81}$$

The required expressions for the conditional moments and the corresponding derivatives have already been determined in Sect. 4.2.1.4. Thus, we only have to insert the terms (4.37)–(4.47) into (4.81), which can easily be calculated with standard computer applications.

4.3.3 Moment Matching Procedure for Stochastic LGDs

Subsequently, we will study the accuracy of the ASRF formula and of the granularity adjustment for the risk measure ES in order to compare the capability of measuring name concentrations in comparison with the VaR (cf. Sect. 4.2.2). However, before we perform the corresponding numerical analyses, we deal with the modeling of stochastic LGDs. Based on this, we can perform our numerical analyses of the ES-based formulas not only for constant LGDs[217] but also for stochastic LGDs. This will show to which degree the accuracy of the ASRF framework and of the granularity adjustments are affected by this additional source of uncertainty. In order to incorporate a realistic degree of uncertainty, the probability distribution of LGDs will not be chosen on an ad-hoc basis, but different density functions will be parameterized in a way that mean and standard deviation

[217] Even if the calculations were based on the portfolio gross loss and thus on an LGD of 100%, the results remain identically for every constant LGD as the numerator and the denominator of the analyzed expressions are affected to the same degree.

will agree with empirical data reported by Schuermann (2005). These density functions, which are typically mentioned in the literature for modeling LGDs, are a normal distribution, a log-normal distribution, a logit-distribution, and a beta-distribution. This moment-matching procedure will be performed for senior secured, senior unsecured, senior subordinated, subordinated, as well as junior subordinated loans. As a next step, the 25%-, 50%-, and 75%-quantiles will be calculated for each of the parameterized distributions. Finally, the distribution with the smallest averaged difference between the calculated and the empirical quantiles will be chosen for the numerical analyses using the parameter setting for senior unsecured loans.

A typical shape of a recovery-rate-distribution, which is the distribution of $1-\mathrm{LGD}$, can be seen in Fig. 4.8. The presented recovery rates correspond to 2,023 defaulted corporate bonds and loans from Moody's Default Risk Service Database. Approximately 88% of these instruments were issued by corporations domiciled in the United States.[218] In the presented case, the distribution is right-skewed, which means that there are many defaults with rather low recovery rates and few defaults with high recovery rates. While in most cases the recovery rate is between 0 and 100%, it is not necessarily bounded between these values. The demonstrated recovery rates of more than 100% appear if the interest rate at the time of recovery is lower than the coupon rate.[219] As mentioned in Sect. 2.2.1,

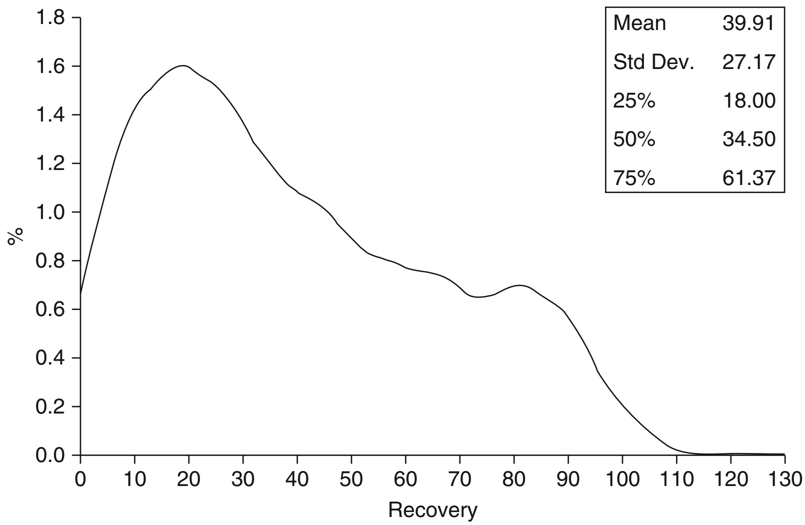

Fig. 4.8 Probability distribution of recovery rates for corporate bonds and loans, 1970–2003. See Schuermann (2005), p. 14

[218]Cf. Schuermann (2005), p. 22, footnote 8.

[219]Cf. Schuermann (2005), p. 22, footnote 11.

the case of recovery rates below 0% can occur due to workout costs. Since the attempt to recover a (part of a) loan is costly, the recovery rate is lower than 0% if the recovery cash flows are smaller than the workout costs. Even if this case is not presented in Fig. 4.8, it is practically more relevant than recovery rates of more than 100% as workout costs always occur whereas the other effect is if at all unsystematic.[220] Nonetheless, the mass of the distribution is between 0 and 100%, so that it can be beneficial to choose a probability distribution which is bounded between these values.

In the literature, there are different proposals for the choice of an LGD distribution. In the context of modeling LGDs that depend on a systematic factor,[221] Frye (2000) used the normal distribution. One point of criticism is that this distribution is symmetric and cannot describe the typically skewed LGDs. Against this background, Pykhtin (2003) chose the lognormal distribution. Schönbucher (2003) applied the logit-normal distribution, which is bounded between 0 and 1. As mentioned above, LGDs do not necessarily fulfill this characteristic but the distribution can almost be seen as bounded in this interval. A further common LGD distribution that is bounded in this interval is the beta distribution,[222] which is for example used in CreditMetrics™.[223] All of these distributions depend on two parameters. Thus, we can parameterize all of these distributions by matching the first two moments with the empirical distribution.

The probability density function of a *normally distributed* random variable $\tilde{X}$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{4.82}$$

with mean μ and standard deviation σ, that is $\tilde{X} \sim \mathcal{N}(\mu, \sigma^2)$. The quantiles q_α of a normal distribution with parameters μ and σ can be calculated as

$$\begin{aligned} \mathbb{P}\left(\tilde{X} \leq q_\alpha\right) &= \Phi\left(\frac{q_\alpha - \mu}{\sigma}\right) = \alpha \\ &\Leftrightarrow \frac{q_\alpha - \mu}{\sigma} = \Phi^{-1}(\alpha) \\ &\Leftrightarrow q_\alpha = \mu + \sigma \cdot \Phi^{-1}(\alpha). \end{aligned} \tag{4.83}$$

[220]Probably, the data used to generate the figure did not include workout costs and therefore underestimate the true economic loss. Furthermore, the choice of the discount rate influences the effect of negative LGDs: If the recovery cash flows are discounted by the contractual rate, as required by IFRS and as proposed by the Basel II framework, a complete recovery without workout costs leads to a recovery rate of 100%, which shows that negative LGDs are not relevant at all.

[221]The issue of interconnections between LGDs and PDs via a systematic factor is not in the scope of this analysis.

[222]Cf. Altman et al. (2005), p. 46.

[223]Cf. Gupton et al. (1997), p. 80.

If a random variable $\tilde{X}$ is normally distributed with $\tilde{X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$, the transformation $\tilde{Y} = e^{\tilde{X}}$ leads to a *lognormally distributed* variable $\tilde{Y}$.[224] The density function is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}y} \exp\left(-\frac{(\ln y - \mu_X)^2}{2\sigma_X^2}\right). \tag{4.84}$$

In order to parameterize the distribution, the parameters μ_X and σ_X have to be expressed as a function of the known mean μ and standard deviation σ. Using the well-known moments of a lognormal distribution[225]

$$\mu = \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right) \quad \text{and} \quad \sigma^2 = (\exp(\sigma_X^2) - 1) \cdot \exp(2\mu_X + \sigma_X^2), \tag{4.85}$$

we obtain

$$\begin{aligned} \sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \exp\left(2\mu_X + \sigma_X^2\right) \\ \Leftrightarrow \sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)^2 \\ \Leftrightarrow \sigma^2 &= \left(\exp\left(\sigma_X^2\right) - 1\right) \cdot \mu^2 \\ \Leftrightarrow \sigma_X^2 &= \ln\left(\frac{\sigma^2}{\mu^2} + 1\right) \end{aligned} \tag{4.86}$$

and

$$\begin{aligned} \mu &= \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right) \\ \Leftrightarrow \mu_X &= \ln\mu - \frac{1}{2}\sigma_X^2 \\ \Leftrightarrow \mu_X &= \ln\mu - \frac{1}{2}\ln\left(\frac{\sigma^2}{\mu^2} + 1\right). \end{aligned} \tag{4.87}$$

As the logarithm of a lognormally distributed variable is normally distributed with mean μ_X and standard deviation σ_X, the cumulative distribution function $F(y)$ can be expressed in terms of the standard normal distribution:

$$F_Y(y) = \Phi\left(\frac{\ln y - \mu_X}{\sigma_X}\right). \tag{4.88}$$

[224] See also Sect. 2.3.

[225] Cf. Bronshtein et al. (2007), p. 760, (16.80).

Similar to (4.83), this leads to

$$\begin{aligned} &\Phi\left(\frac{\ln q_\alpha - \mu_X}{\sigma_X}\right) = \alpha \\ \Leftrightarrow &q_\alpha = \exp\left(\mu_X + \sigma_X \cdot \Phi^{-1}(\alpha)\right). \end{aligned} \tag{4.89}$$

A *logit-normal distribution* results from a normally distributed variable $\tilde{X}$ with $\tilde{X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$, which is transformed by the logit function $\tilde{Y} = e^{\tilde{X}} / (1 + e^{\tilde{X}})$. The transformation assures that the transformed variable is bounded to [0, 1]. As shown in Appendix 4.5.16, the probability density function is given as

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(\ln(1/y - 1) + \mu_X)^2}{2\sigma_X^2}\right) \frac{1}{y(1-y)}. \tag{4.90}$$

Since an analytical determination of mean and standard deviation is not obvious, the parameterization will be done numerically. For this purpose, the moments will be computed for different μ_X/σ_X-combinations until the deviation of both parameters from the empirical data is less than 10^{-4}. The corresponding quantiles will be determined via numerical integration of (4.90).

The density of a *beta distribution* with shape parameters $\alpha, \beta > 0$ can be defined as

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \tag{4.91}$$

where the beta function $B(\alpha, \beta)$ is defined as

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \tag{4.92}$$

or as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.93}$$

using the gamma function $\Gamma(\cdot)$.[226] With mean and variance

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}, \tag{4.94}$$

[226]Cf. Schönbucher (2003), p. 147 f.

the beta distribution can be parameterized using the following shape parameters

$$\mu = \frac{\alpha}{\alpha + \beta},$$
$$\Leftrightarrow \beta = \frac{\alpha}{\mu} - \alpha, \tag{4.95}$$

and

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$$
$$\Leftrightarrow \sigma^2 = \frac{\alpha^2(1/\mu - 1)}{(\alpha/\mu)^2(1+\alpha/\mu)}$$
$$\Leftrightarrow \sigma^2 = \frac{\mu^2(1-\mu)}{(\mu+\alpha)}$$
$$\Leftrightarrow \alpha = \frac{\mu^2(1-\mu)}{\sigma^2} - \mu. \tag{4.96}$$

Similar to the logit-normal distribution, the quantiles of the beta distribution will be determined via numerical integration of (4.91).

As mentioned above, the different distribution functions will be parameterized using the data for corporate bonds and loans reported by Schuermann (2005). These data contain information about the empirical mean and standard deviation as well as the 25%-, 50%-, 75%-quantiles, and the number of observations N of recovery rates for different seniorities (see Table 4.8).[227] As expected, the average recovery rate as well as the quantiles of the recovery rate distribution are mostly the higher, the more senior the debt instrument.

In Tables 4.9–4.12, the determined parameters, which lead to a matching of moments, of the four considered distributions are reported for each of the seniorities. Furthermore, the corresponding quantiles $\hat{q}$ that result for these distributions are reported in the respective tables. The root mean squared errors (RMSE) are

Table 4.8 Recovery rates by seniority, 1970–2003[a]

Seniority	Mean μ	Std. dev. σ	$q_{0.25}$ (%)	$q_{0.5}$ (%)	$q_{0.75}$ (%)	N
Senior secured	0.543	0.258	33.00	53.50	75.00	433
Senior unsecured	0.387	0.278	14.50	30.75	63.00	971
Senior subordinated	0.285	0.234	10.00	23.00	42.25	260
Subordinated	0.347	0.222	19.50	30.29	45.25	347
Junior subordinated	0.144	0.090	9.13	13.00	19.13	12

[a]See Schuermann (2005), p. 16

[227]The aggregated data correspond to Fig. 4.8.

Table 4.9 Results of the normal distribution

Seniority	μ	σ	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	0.543	0.258	36.84	54.26	71.68	2.97
Senior unsecured	0.387	0.278	19.96	38.71	57.46	6.43
Senior subordinated	0.285	0.234	12.72	28.51	44.30	3.74
Subordinated	0.347	0.222	19.66	34.65	49.64	3.57
Junior subordinated	0.144	0.090	8.33	14.39	20.45	1.20
						Ø 3.58

Table 4.10 Results of the lognormal distribution

Seniority	μ_X	σ_X	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	−0.713	0.452	36.13	49.00	66.45	5.86
Senior unsecured	−1.157	0.645	20.35	31.44	48.58	9.00
Senior subordinated	−1.513	0.718	13.58	22.03	35.76	4.32
Subordinated	−1.232	0.587	19.63	29.16	43.33	1.28
Junior subordinated	−2.103	0.574	8.29	12.20	17.97	0.95
						Ø 4.28

Table 4.11 Results of the logit-normal distribution

Seniority	μ_X	σ_X	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	0.234	1.396	33.02	55.82	76.41	1.57
Senior unsecured	−0.686	1.679	13.98	33.51	60.99	1.99
Senior subordinated	−1.284	1.493	9.20	21.70	43.13	1.02
Subordinated	−0.819	1.224	16.20	30.61	50.17	3.43
Junior subordinated	−1.967	0.741	7.83	12.28	18.75	0.89
						Ø 1.78

Table 4.12 Results of the beta distribution

Seniority	α	β	$\hat{q}_{0.25}$ (%)	$\hat{q}_{0.5}$ (%)	$\hat{q}_{0.75}$ (%)	RMSE (%)
Senior secured	1.477	1.245	33.59	55.43	75.84	1.26
Senior unsecured	0.801	1.269	14.04	34.58	60.63	2.61
Senior subordinated	0.775	1.944	8.61	22.85	44.01	1.30
Subordinated	1.241	2.341	16.27	31.55	50.37	3.57
Junior Subordinated	2.050	12.193	7.55	12.72	19.50	0.95
						Ø 1.94

reported as a quality criterion of the accuracy of the estimated quantiles in comparison with the empirical quantiles:

$$\mathrm{RMSE} = \sqrt{\frac{1}{3}\left[(\hat{q}_{0.25} - q_{0.25})^2 + (\hat{q}_{0.5} - q_{0.5})^2 + (\hat{q}_{0.75} - q_{0.75})^2\right]}. \tag{4.97}$$

Finally, the averaged RMSE is reported for every distribution in order to determine the most appropriate description of an LGD distribution.

As can be seen from the tables, the normal and the lognormal distribution cannot fit the empirical data very well. By contrast, both the parameterized logit-normal and the beta distribution lead to a good accuracy with respect to the considered quantiles. As the logit-normal distribution leads to the smallest averaged RMSE, this distribution will be used to analyze the accuracy of the ASRF solution and the granularity adjustments for stochastic LGDs. For this purpose, the moments and the determined parameter values for *senior unsecured* bonds and loans will be implemented.

4.3.4 Numerical Analysis of the ES-Based Granularity Adjustment

4.3.4.1 Impact on the Portfolio-Quantile

In Sect. 4.2.2, we have studied the accuracy of the ASRF formula and the granularity adjustment for the risk measure VaR. However, we do not know how good the ES-based measurement of portfolio name concentration risk performs in comparison to the VaR-based measurement. Thus, our preceding analyses will be performed for the coherent ES subsequently. Moreover, we test the impact of stochastic LGDs on the accuracy of our approximation formulas. We start with an analysis of:

(a) The numerically “exact” coarse grained solution (see (2.76))
(b) The fine grained ASRF solution (see (4.59))
(c) The ASRF solution with first-order adjustment (see (4.70) and (4.73))
(d) The ASRF solution with first- and second-order adjustments (see (4.78) and (4.81))

for a homogeneous portfolio consisting of 40 credits with $PD = 1\%, LGD = 100\%$, and $\rho = 20\%$. The resulting ES using the formulas for the “exact” solution (a) as well as approximations (b) to (d) is presented in Fig. 4.9 for confidence levels starting at 0.7. In Fig. 4.10, the results for high confidence levels from 0.994 on are shown.

As can be seen in the figures, the ASRF solution underestimates the risk because the idiosyncratic component is neglected. Especially for high confidence levels, the impact of this underestimation is very high. The first-order granularity adjustment seems to be a very good approximation for a broad range of confidence levels. If the figures corresponding to the ES are compared to those of the VaR (see Figs. 4.1 and 4.2), the adjustment formula using the ES seems to work even better than the formula using the VaR. Unfortunately, it seems that the second-order adjustment cannot improve the result. Even if the approximation for high confidence levels is very good, the accuracy for lower confidence levels is significantly lower than without this additional adjustment.

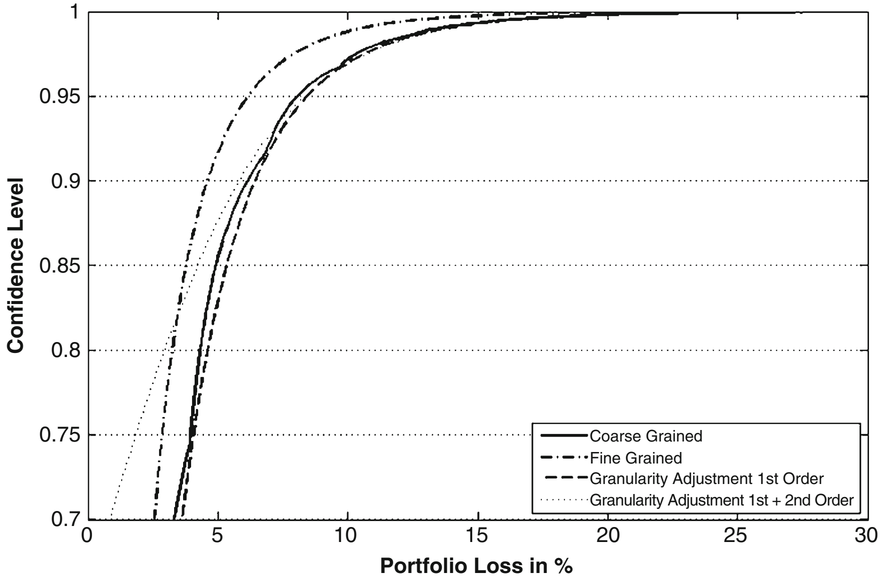

Fig. 4.9 Expected Shortfall for a wide range of probabilities

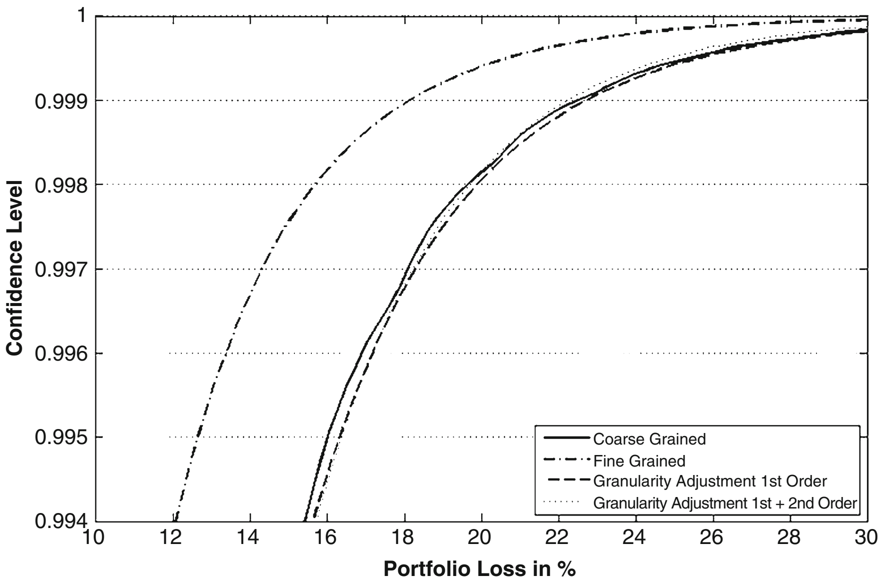

Fig. 4.10 Expected Shortfall for high confidence levels

In order to get a better insight in the accuracy of the different approximations, subsequently several numerical analyses will be performed similar to Sects. 4.2.2.2–4.2.2.4. In these sections, we have defined two kinds of critical numbers. The first measured the minimum number of credits a portfolio must consist of to have a good approximation of the "true" VaR at confidence level 0.999. The second number measured the critical number of credits for which the ASRF approximation of the 99.9%-VaR does not exceed the VaR at confidence level 0.995. Assuming that the increase of the confidence level from 0.995 to 0.999 happened to compensate the neglect of the granularity adjustment, it can be argued that the idiosyncratic risk component is already accounted for if the resulting critical number of credits is exceeded, whereas for a lower number of credits the risk is underestimated (for an actually intended confidence level of 0.995). The first type of analysis directly tests the performance of the different approaches. On the contrary, the second type of analysis does not focus on the accuracy of the approximation formulas but analyzes the need of additional economic capital against the specific regulatory setting. Thus, in order to test the performance of the different approximation formulas when using a different risk measure, only the first type of analyses will be performed in the following.[228] Due to the changed risk measure, the true risk will be given by the 99.72%-ES within the Vasicek model instead of the 99.9%-VaR.[229]

4.3.4.2 Size of Fine Grained Risk Buckets

Similar to Sect. 4.2.2.2, it will be determined for which portfolios the ES-based ASRF solution is a good approximation of the "true" ES. This will be done with a target tolerance of $\beta = 5\%$:[230]

$$I_{c,\mathrm{ES},\mathrm{det.}}^{(\mathrm{ASRF})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\mathrm{ASRF})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{n}\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \right) \text{ with } \beta = 0.05. \quad (4.98)$$

[228] The critical number of credits in a portfolio which leads to equality of the different parameter settings of the Basel consultative documents is not of interest in the subsequent analyses regarding the ES as both rely on the VaR.

[229] See Sect. 4.3.1.

[230] As the ASRF solution is constant and the coarse grained solution is monotonously decreasing in n for the ES (this is a result of the monotonicity of specific risk-property, cf. Sect. 4.3.1), the inequality also holds for every number above the first number that satisfies the inequality. Thus, the expression "for all $N \geq n$", which had to be included in the corresponding analysis for the VaR, can be neglected.

Table 4.13 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.98))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	30,405	20,112	4,711	4,516	3,593	2,803	1,828	1,246	893	443	346	191
3.5%	25,215	16,766	3,996	3,815	3,048	2,399	1,571	1,077	775	389	306	171
4.0%	21,425	14,273	3,460	3,297	2,644	2,079	1,375	946	686	348	275	155
4.5%	18,300	12,267	3,022	2,883	2,319	1,829	1,213	844	612	315	249	142
5.0%	15,920	10,714	2,663	2,561	2,054	1,628	1,090	758	553	286	228	132
5.5%	14,044	9,432	2,377	2,290	1,838	1,459	979	685	502	263	210	122
6.0%	12,434	8,443	2,140	2,058	1,658	1,319	889	625	461	243	195	113
6.5%	11,167	7,513	1,944	1,858	1,512	1,208	812	574	425	226	181	106
7.0%	9,985	6,786	1,765	1,701	1,374	1,100	750	529	393	211	170	101
7.5%	9,020	6,163	1,618	1,550	1,265	1,016	689	492	364	198	159	95
8.0%	8,201	5,617	1,490	1,426	1,169	933	641	456	342	186	150	90
8.5%	7,508	5,135	1,378	1,318	1,083	865	598	426	318	175	142	85
9.0%	6,922	4,709	1,277	1,222	1,007	805	555	400	299	166	135	81
9.5%	6,342	4,336	1,186	1,136	937	751	519	376	283	156	128	77
10.0%	5,833	4,054	1,104	1,059	874	702	487	354	267	149	122	74
10.5%	5,455	3,738	1,031	999	816	660	462	334	253	142	116	72
11.0%	5,035	3,462	974	933	764	623	434	315	240	136	111	68
11.5%	4,669	3,259	911	873	719	585	409	298	227	129	106	66
12.0%	4,386	3,021	854	824	681	551	386	283	216	123	102	64
12.5%	4,075	2,860	812	778	640	525	367	268	205	119	98	60
13.0%	3,845	2,657	762	732	611	495	349	257	196	114	94	58
13.5%	3,587	2,524	725	697	575	469	331	244	188	109	90	56
14.0%	3,389	2,351	684	657	545	447	318	233	179	105	87	54
14.5%	3,201	2,237	652	628	519	424	301	224	171	100	83	53
15.0%	3,002	2,095	617	593	493	405	290	213	166	97	80	51
15.5%	2,861	1,991	591	567	470	385	275	205	158	94	78	49
16.0%	2,684	1,905	558	538	452	369	265	196	152	90	75	47
16.5%	2,548	1,782	536	514	428	353	252	189	146	87	72	47
17.0%	2,438	1,703	508	495	411	337	244	181	141	85	71	45
17.5%	2,292	1,634	487	468	391	325	232	175	136	81	68	44
18.0%	2,181	1,532	469	450	375	309	224	167	131	79	66	42
18.5%	2,092	1,467	445	432	362	298	214	162	126	76	64	42
19.0%	1,998	1,411	428	411	344	288	207	155	123	74	62	40
19.5%	1,884	1,330	413	397	332	274	200	150	118	72	60	39
20.0%	1,806	1,273	393	384	321	265	191	146	115	69	59	37
20.5%	1,739	1,225	378	364	306	257	185	140	110	68	57	37
21.0%	1,653	1,182	366	351	295	244	180	136	107	65	56	37
21.5%	1,572	1,114	350	340	286	236	172	132	104	64	54	34
22.0%	1,512	1,070	336	324	273	229	167	126	100	62	53	34
22.5%	1,459	1,032	325	313	263	219	162	123	98	60	51	34
23.0%	1,411	999	315	303	255	212	155	120	94	59	50	32
23.5%	1,329	946	301	294	248	206	151	115	91	57	48	31
24.0%	1,277	908	290	280	237	200	146	112	89	56	47	31

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.14 Critical number of credits from that ASRF solution can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.99))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	44,234	22,604	5,767	5,416	4,464	3,201	2,291	1,517	1,097	585	455	270
3.5%	28,168	20,206	4,362	4,764	3,597	2,615	1,785	1,312	1,022	476	397	245
4.0%	23,449	16,611	4,007	3,838	2,743	2,378	1,665	1,196	806	478	358	220
4.5%	21,337	16,066	3,438	3,592	2,877	2,393	1,423	1,039	855	403	316	207
5.0%	22,141	15,503	3,048	2,993	2,361	1,907	1,313	970	655	375	277	202
5.5%	20,044	11,914	2,600	3,112	2,197	1,497	1,157	794	623	351	265	172
6.0%	14,358	12,750	2,264	2,226	1,820	1,550	1,119	890	598	304	247	172
6.5%	17,261	10,528	2,174	2,283	1,852	1,461	909	637	550	325	248	159
7.0%	11,413	8,966	2,068	1,968	1,649	1,235	864	623	506	261	234	152
7.5%	10,555	10,372	1,718	1,728	1,481	1,379	851	627	506	237	210	149
8.0%	11,789	6,450	1,665	1,554	1,380	1,395	701	624	449	243	206	137
8.5%	11,395	6,049	1,605	1,672	1,307	1,086	651	463	391	227	206	129
9.0%	10,290	5,363	1,689	1,463	1,264	1,201	682	459	372	217	202	130
9.5%	6,833	6,043	1,588	1,432	1,028	853	737	474	373	203	171	121
10.0%	5,945	4,474	1,148	1,404	1,013	1,051	590	443	386	191	157	117
10.5%	8,491	3,458	1,197	1,283	1,012	818	594	462	346	180	157	113
11.0%	8,144	3,707	1,218	999	973	623	593	424	322	178	128	116
11.5%	4,860	3,684	1,066	1,103	752	864	405	376	282	180	145	106
12.0%	5,745	4,733	1,016	1,026	795	918	497	379	252	150	160	108
12.5%	5,918	3,352	1,032	903	756	677	502	315	253	156	133	107
13.0%	3,832	3,041	831	860	734	586	394	342	262	145	116	98
13.5%	4,284	2,810	1,005	884	805	558	397	310	292	149	127	95
14.0%	3,910	2,088	690	884	743	450	327	265	232	134	119	93
14.5%	4,854	3,034	876	683	741	495	428	245	215	132	119	91
15.0%	3,233	2,371	661	684	737	454	446	243	209	130	115	91
15.5%	3,357	3,308	858	551	583	529	323	314	163	126	97	90
16.0%	2,923	2,531	1,039	824	695	449	302	238	186	119	103	86
16.5%	4,623	1,675	630	609	643	416	433	214	182	117	106	84
17.0%	2,413	2,016	759	573	527	493	333	231	214	115	100	84
17.5%	2,406	2,145	517	468	430	384	280	235	190	122	92	82
18.0%	2,465	1,660	588	483	496	356	286	223	167	103	91	86
18.5%	3,963	2,814	600	476	543	436	222	197	144	99	89	80
19.0%	2,040	2,018	462	458	479	348	221	206	156	105	94	79
19.5%	2,533	1,331	421	500	488	320	246	216	154	97	88	76
20.0%	2,763	1,587	419	528	341	323	239	173	142	94	85	78
20.5%	2,408	1,490	535	505	476	354	230	205	163	98	80	77
21.0%	2,819	1,144	354	406	383	271	221	173	158	81	81	78
21.5%	2,106	1,105	380	503	372	227	202	172	125	114	87	75
22.0%	2,748	1,317	401	332	294	281	225	181	140	72	77	74
22.5%	2,709	1,185	450	311	370	249	169	149	127	81	76	71
23.0%	1,579	1,055	452	350	284	263	179	173	103	81	77	71
23.5%	1,785	2,476	384	430	269	258	181	132	148	80	72	71
24.0%	2,399	957	410	330	244	210	167	156	121	85	70	70

Corporates, sovereigns, and banks | SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.) | Mortgage | Revolving retail | Other retail

Moreover, we measure the accuracy of the ASRF solution if LGDs are stochastic and following a logit-normal distribution with

$$I_{c,\text{ES,stoch.}}^{(\text{ASRF})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\text{ASRF})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \right) \text{with } \beta = 0.05. \tag{4.99}$$

In contrast to the analyses of Sect. 4.2.2 and the preceding definition of a critical number for deterministic LGDs (4.98), the denominator, which is the benchmark for the ASRF solution, cannot be determined with the Vasicek model because it does not account for stochastic LGDs. Against this background, we perform Monte Carlo simulations with one million trials for each PD/ρ-combination and for every number of credits until the target accuracy is reached.

The resulting critical numbers for the case of deterministic LGDs $I_{c,\text{ES,det.}}^{(\text{ASRF})}$ are reported in Table 4.13 for a broad range of correlations and PDs. Similar to the corresponding VaR-analysis, the values $I_{c,\text{ES,det.}}^{(\text{ASRF})}$ vary from 31 for a high PD/ρ-combination to 30,405 for a low PD/ρ-combination. This shows that at least for non-retail portfolios the assumption of infinite granularity is critical for real-world portfolios and the chosen risk measure does not influence the accuracy of the ASRF solution to a great extent.

The corresponding critical numbers for stochastic LGDs $I_{c,\text{ES,stoch.}}^{(\text{ASRF})}$ are reported in Table 4.14. As expected, the accuracy of the ASRF solution is lower for stochastic than for deterministic LGDs because there is an additional source of unsystematic uncertainty. In comparison with the case of deterministic LGDs, the minimum number of credits increased from a range between 31 and 31,405 to a range between 70 and 44,234 credits. On average, the required portfolio size is 31.55% higher due to stochastic LGDs if the identical accuracy shall be achieved.

4.3.4.3 Probing First-Order Granularity Adjustment

In order to test the accuracy of the ES-based first-order granularity adjustment, we determine the critical number $I_{c,\text{ES,det.}}^{(\text{1st Order Adj.})}$, which is the minimum number of credits to deliver a good approximation of the "true" ES on a 99.72% confidence level, for different PD/ρ-combinations. These critical values

$$I_{c,\text{ES,det.}}^{(\text{1st Order Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(\text{1st Order Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right), \text{with } \beta = 0.05, \tag{4.100}$$

are presented in Table 4.15. As the ES-based first-order granularity adjustment does not only take the conditional variance of the default indicator into account but also the second moment of LGDs, it is interesting to find out how good the granularity adjustment performs in the presence of stochastic LGDs. For this purpose, we also determine the critical values

$$I_{c,\mathrm{ES,stoch.}}^{(1.\,\mathrm{Order\,Adj.})} = \inf\left(n : \left| \frac{ES_{0.9972}^{(1.\,\mathrm{Order\,Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N}\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right)} - 1 \right| < \beta \;\; \forall N \in \mathbb{N}^{\geq n} \right) \tag{4.101}$$

with $\beta = 0.05$, which are shown in Table 4.16.

For deterministic LGDs, the minimum number of credits varies between 7 and 2,468, which is a reduction of averaged 91.64% compared to the ASRF solution (see Table 4.13 in Sect. 4.3.4.2). Thus, we have a significant improvement of the accuracy if the first-order adjustment is taken into account. A very interesting finding results if the accuracy of the granularity adjustment is compared for the VaR and the ES. Even for a portfolio that consists of averaged 49.05% less credits and thus contains significantly more idiosyncratic risk, we are able to achieve the identical accuracy if name concentrations are measured on the basis of the Expected Shortfall instead of the Value at Risk. For the most relevant cases, where the minimum portfolio size is relatively high, this effect is even stronger.

If the improvement is analyzed only for cases where the minimum portfolio size is higher than 100 credits (determined for the VaR-based granularity adjustment), we find that the target accuracy can still be achieved if the portfolio consists of 68.91% less portfolios compared to a VaR-based measurement. For example, a high quality retail portfolio (AAA) must consist of at least 1,588 credits instead of 5,027 credits if name concentration is measured with the ES. Similarly, a medium quality corporate portfolio (BBB) must contain 25 compared to 106 credits. This shows that the already good performance of the VaR-based granularity adjustment can be improved significantly if name concentrations are measured with the ES.

The results for stochastic LGDs, which are presented in Table 4.16, are very promising. In most cases, the accuracy is slightly higher than in the case of deterministic LGDs. On average, the required portfolio size is reduced by 3.64%. Concretely, the accuracy is higher/identical/lower for 272/35/209 elements of the matrix. Of course, the results are influenced by a small degree of simulation noise but the accuracy seems to be at least identically in the presence of stochastic LGDs. If the accuracy of the granularity adjustment is compared with the ASRF solution of Table 4.14, the minimum number of credits is about 92.19% lower,[231] which is an excellent result. As a further robustness check, the corresponding values are determined for beta-distributed LGDs. In this case, the

[231]The corresponding value for deterministic LGDs is 91.64%.

Table 4.15 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.100))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	2,468	1,870	439	466	367	279	192	148	111	62	53	38
3.5%	2,198	1,410	396	377	294	223	157	125	94	55	45	34
4.0%	1,588	1,010	313	298	266	205	145	106	81	48	40	29
4.5%	1,453	930	287	274	214	186	119	89	69	42	36	27
5.0%	976	858	224	213	198	152	111	83	64	37	34	25
5.5%	911	792	209	199	155	142	90	69	55	33	30	24
6.0%	853	726	195	186	146	112	85	65	52	31	27	22
6.5%	800	514	147	173	138	106	80	61	44	30	26	20
7.0%	752	485	139	133	129	100	64	51	42	27	23	20
7.5%	707	458	132	126	99	95	61	49	40	26	23	18
8.0%	665	433	126	120	94	89	58	47	33	25	22	18
8.5%	625	410	120	114	90	70	56	44	32	22	19	17
9.0%	585	250	113	108	86	67	53	36	31	21	19	16
9.5%	540	240	107	103	82	64	51	35	30	20	18	16
10.0%	358	231	101	74	79	62	40	34	29	20	16	13
10.5%	343	222	75	72	75	59	38	33	28	17	16	13
11.0%	330	213	72	69	71	57	37	25	23	17	16	13
11.5%	317	206	70	67	53	54	36	24	22	16	13	13
12.0%	305	198	67	64	51	52	35	24	22	16	13	13
12.5%	294	191	65	62	49	50	34	23	21	16	13	13
13.0%	283	185	63	60	48	37	33	23	20	13	13	11
13.5%	273	178	61	58	46	36	32	22	20	13	13	11
14.0%	264	172	59	56	45	35	31	21	19	13	12	11
14.5%	120	167	57	54	44	34	30	21	19	13	12	11
15.0%	117	161	55	53	42	33	29	20	18	12	12	11
15.5%	114	156	53	51	41	32	28	20	18	12	12	11
16.0%	111	151	51	33	40	31	26	19	14	12	12	11
16.5%	109	147	33	32	39	31	20	19	14	12	10	11
17.0%	106	142	33	31	37	30	20	18	14	12	10	11
17.5%	104	138	32	30	36	29	19	18	14	11	10	11
18.0%	101	134	31	30	35	28	19	18	13	11	10	11
18.5%	99	130	30	29	34	27	19	13	13	9	10	8
19.0%	97	63	30	28	23	27	18	13	13	9	9	9
19.5%	95	61	29	28	22	17	18	13	9	9	9	9
20.0%	93	60	28	27	22	17	17	12	9	9	9	9
20.5%	91	59	28	27	21	17	17	12	9	9	9	9
21.0%	89	58	27	26	21	16	17	12	9	9	9	9
21.5%	88	57	27	25	20	16	16	12	9	9	9	9
22.0%	86	56	26	25	20	16	16	11	9	9	7	9
22.5%	84	55	26	24	20	16	16	11	11	9	7	9
23.0%	83	54	25	24	19	15	15	11	11	8	7	9
23.5%	81	53	25	23	19	15	15	11	11	8	7	9
24.0%	80	52	24	23	19	15	15	11	11	8	7	9

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

Table 4.16 Critical number of credits from that the first order adjustment can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.101))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	2,338	1,682	531	470	403	308	243	158	126	74	66	50
3.5%	1,745	1,371	367	360	308	226	190	130	103	63	54	42
4.0%	1,663	1,104	315	308	241	214	151	117	89	54	49	39
4.5%	1,272	906	259	248	204	171	132	92	74	49	43	36
5.0%	1,055	779	225	224	175	164	112	89	64	41	37	33
5.5%	841	575	179	207	151	123	91	68	55	38	36	31
6.0%	758	506	165	158	140	107	85	70	53	35	33	29
6.5%	620	436	145	142	124	106	81	64	50	33	30	26
7.0%	595	416	126	122	129	94	63	63	46	30	27	26
7.5%	515	346	111	119	96	79	64	48	41	27	25	24
8.0%	473	335	101	107	89	72	61	42	36	25	25	23
8.5%	415	327	89	89	77	71	52	37	32	23	23	23
9.0%	272	290	79	86	75	66	48	38	32	23	22	21
9.5%	269	163	72	75	62	57	47	38	30	22	20	21
10.0%	233	170	74	69	64	56	36	32	27	20	19	19
10.5%	221	146	67	61	60	52	38	28	27	21	19	19
11.0%	189	146	64	60	58	50	34	35	25	20	18	19
11.5%	191	127	56	58	46	49	35	26	24	17	17	18
12.0%	174	119	56	54	45	35	35	23	23	18	16	17
12.5%	180	113	54	51	41	34	29	23	22	16	16	17
13.0%	169	111	51	48	37	31	30	22	22	15	14	16
13.5%	163	106	54	41	41	33	25	21	20	15	14	17
14.0%	142	102	42	41	35	33	23	22	19	15	14	15
14.5%	151	98	42	46	33	30	20	18	17	13	14	16
15.0%	139	92	42	44	30	28	25	18	16	12	13	16
15.5%	137	89	31	37	32	27	18	16	15	13	13	16
16.0%	133	89	45	36	31	27	19	16	15	12	13	15
16.5%	125	87	26	29	30	24	18	16	14	13	12	14
17.0%	131	79	36	24	20	23	17	16	13	11	12	14
17.5%	119	81	21	31	26	24	18	13	14	11	12	15
18.0%	105	81	21	23	25	22	15	12	13	11	11	15
18.5%	122	80	20	22	19	21	15	12	13	10	11	15
19.0%	109	77	21	19	16	17	15	11	12	11	10	14
19.5%	115	80	20	19	17	17	15	12	11	10	10	15
20.0%	112	69	18	18	15	17	15	11	11	10	10	14
20.5%	105	71	18	17	25	19	15	10	10	9	10	14
21.0%	102	69	17	15	14	16	14	10	10	10	9	14
21.5%	101	62	17	16	14	14	12	13	9	9	9	14
22.0%	92	62	17	15	13	14	14	10	8	8	9	13
22.5%	88	63	16	14	13	10	12	10	10	9	10	14
23.0%	86	67	15	14	12	14	11	10	9	9	9	14
23.5%	83	59	15	15	13	11	10	9	9	8	8	14
24.0%	97	58	14	15	12	10	12	9	9	8	8	14

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

target accuracy is already reached for 4.89% less credits, compared to the case of deterministic LGDs. In comparison to the ASRF solution, the critical number is 92.27% lower.

4.3.4.4 Probing Second-Order Granularity Adjustment

As a next step, we analyze the accuracy of the ES-based second-order adjustment in comparison to the "exact" ES for deterministic LGDs:

$$I_{c,\mathrm{ES,det.}}^{(\text{1st}+\text{2nd Order Adj.})} = \inf\left(n : \left|\frac{ES_{0.9972}^{(\text{1st}+\text{2nd Order Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tilde{D}_i\}}\right)} - 1\right| < \beta \ \forall N \in \mathbb{N}^{\geq n}\right) \tag{4.102}$$

with $\beta = 0.05$. Moreover, the second order granularity adjustment is tested for stochastic LGDs using the formula

$$I_{c,\mathrm{ES,stoch.}}^{(1.+2.\text{ Order Adj.})} = \inf\left(n : \left|\frac{ES_{0.9972}^{(1.+2.\text{ Order Adj.})}\left(\tilde{L}\right)}{ES_{0.9972}^{(n)}\left(\tilde{L} = \frac{1}{N}\sum_{i=1}^{N} \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}\right)} - 1\right| < \beta \ \ \forall N \in \mathbb{N}^{\geq n}\right) \tag{4.103}$$

with $\beta = 0.05$. Due to the second-order adjustment, not only the variance but also the skewness of LGDs is accounted for in the approximation formula.

The results for deterministic LGDs, which are reported in Table 4.17, confirm the findings of Fig. 4.9 and also of the corresponding VaR-based analysis of Sect. 4.2.2.4. If concentration risk is measured with the second-order adjustment, the required portfolio size is 89.79% smaller than without the adjustment formula and it performs still better than the VaR-based adjustment formulas but there is no improvement compared to the ES-based first-order adjustment. Thus, it has to be stated that the second-order adjustment formula stemming from additional elements of the Taylor series expansion is performing worse than the first-order adjustment. As discussed in Sect. 4.2.2.4, it remains unclear if this unexpected result is e.g. a consequence of a non-converging Taylor series or if the consideration of more elements of the Taylor series could improve the approximation. But for all that, we found that the ES-based first-order adjustment is an excellent method for measuring name concentrations.

The corresponding results for stochastic LGDs are reported in Table 4.18. Interestingly, the results for low PDs and high correlation parameters are very good, whereas for high PDs and low correlation parameters the results are worse

Table 4.17 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true ES if LGDs are deterministic (see (4.102))

	AAA up to AA− 0.03%	A− up to A+ 0.05%	BBB+ 0.32%	BBB 0.34%	BBB− 0.46%	BB+ 0.64%	BB 1.15%	BB− 1.97%	B+ 3.19%	B 8.99%	B− 13.01%	CCC up to C 30.85%
3.0%	3,381	2,533	880	841	707	585	433	338	271	178	159	131
3.5%	2,036	1,627	663	634	542	454	347	270	222	151	135	114
4.0%	1,302	1,127	491	473	413	355	279	223	183	130	118	103
4.5%	760	741	389	374	333	289	226	185	156	115	105	94
5.0%	594	443	306	295	269	237	189	159	136	102	94	86
5.5%	256	238	237	229	215	194	160	138	120	91	85	80
6.0%	466	161	180	176	169	157	135	120	107	84	78	74
6.5%	473	273	159	153	152	129	123	105	95	75	72	70
7.0%	746	453	113	110	116	113	103	91	84	69	67	66
7.5%	722	447	101	98	87	89	86	80	75	64	63	63
8.0%	695	435	66	65	76	80	80	73	67	60	59	59
8.5%	668	421	58	56	69	61	65	64	63	56	55	57
9.0%	641	407	33	50	46	54	61	59	56	52	52	55
9.5%	614	392	27	27	41	50	50	56	53	50	50	53
10.0%	588	378	23	23	37	35	45	48	50	47	47	51
10.5%	563	363	39	36	34	31	42	45	44	45	45	49
11.0%	539	350	40	38	18	28	40	43	42	42	43	48
11.5%	515	336	41	38	16	26	31	36	38	41	42	47
12.0%	492	323	64	60	14	15	29	34	36	38	39	45
12.5%	469	310	63	59	27	13	27	33	34	37	38	44
13.0%	445	298	62	59	27	12	26	28	33	36	37	43
13.5%	420	286	61	58	27	11	18	26	29	34	35	42
14.0%	292	274	60	56	42	19	17	25	28	33	35	42
14.5%	282	262	58	55	42	19	16	24	27	32	34	40
15.0%	272	178	57	54	41	19	15	23	26	31	32	40
15.5%	263	173	56	53	41	19	14	22	25	30	31	39
16.0%	254	168	54	52	40	29	9	18	25	29	31	38
16.5%	245	162	53	33	39	29	8	17	21	28	30	38
17.0%	237	158	52	33	38	28	8	16	21	27	30	37
17.5%	229	153	51	48	38	28	7	16	20	27	28	37
18.0%	221	148	33	47	37	28	7	15	19	26	28	37
18.5%	213	144	48	46	36	27	7	15	19	26	27	36
19.0%	206	139	47	45	36	27	6	14	18	25	27	35
19.5%	198	135	46	44	35	26	6	14	18	25	26	35
20.0%	191	131	45	43	34	17	6	14	18	23	26	35
20.5%	183	127	44	42	33	17	3	10	17	23	26	34
21.0%	176	123	43	41	33	17	3	10	15	23	25	34
21.5%	91	62	42	40	32	17	3	9	15	22	25	34
22.0%	88	60	41	39	31	16	4	9	14	22	25	34
22.5%	86	58	40	39	31	16	4	9	14	22	25	34
23.0%	83	57	39	38	30	23	4	9	14	21	23	33
23.5%	81	56	38	37	30	23	4	8	13	21	23	33
24.0%	78	54	37	36	29	23	4	8	13	21	23	33

Corporates, sovereigns, and banks | SMEs (5Mio. < Sales < 50 Mio.)
SMEs (Sales < 5 Mio.) | Mortgage | Revolving retail | Other retail

Table 4.18 Critical number of credits from that the first plus second order adjustment can be stated to be sufficient for measuring the true ES if LGDs are stochastic (see (4.103))

	AAA up to AA−	A− up to A+	BBB+	BBB	BBB−	BB+	BB	BB−	B+	B	B−	CCC up to C
	0.03%	0.05%	0.32%	0.34%	0.46%	0.64%	1.15%	1.97%	3.19%	8.99%	13.01%	30.85%
3.0%	4,175	3,045	1,045	980	818	699	499	393	327	227	201	181
3.5%	2,745	2,102	835	761	674	546	435	323	272	190	167	154
4.0%	1,699	1,410	618	579	548	424	331	275	218	165	148	144
4.5%	1,090	951	477	462	419	361	282	230	194	140	135	131
5.0%	541	632	398	396	347	272	252	184	163	128	119	120
5.5%	264	347	311	287	277	256	197	170	144	113	110	110
6.0%	288	210	254	258	210	198	162	136	130	105	96	104
6.5%	600	136	203	193	178	164	142	124	113	96	89	98
7.0%	652	388	158	159	137	139	131	105	101	84	82	92
7.5%	670	358	126	115	126	116	112	102	89	81	81	87
8.0%	670	376	95	93	103	108	91	86	90	75	72	85
8.5%	613	408	73	75	81	84	89	85	81	70	69	80
9.0%	555	368	47	46	64	70	77	73	72	67	65	80
9.5%	575	316	37	36	55	59	63	65	63	62	62	76
10.0%	531	364	24	29	38	48	62	63	61	60	61	75
10.5%	550	321	11	12	31	41	55	60	53	54	57	71
11.0%	495	323	35	18	23	30	46	45	51	53	55	70
11.5%	431	276	47	46	11	24	40	46	45	52	53	69
12.0%	366	278	54	49	8	22	34	44	44	49	51	69
12.5%	428	295	55	51	15	18	32	41	41	46	49	65
13.0%	424	271	55	50	18	16	27	37	36	45	47	65
13.5%	367	264	63	46	37	7	26	37	38	42	47	63
14.0%	225	233	52	49	34	6	24	31	34	41	46	65
14.5%	333	227	53	61	35	10	22	29	31	44	42	62
15.0%	215	220	54	53	35	24	16	27	31	40	42	63
15.5%	204	193	56	49	36	21	17	26	30	37	41	60
16.0%	191	189	54	46	36	25	13	24	28	37	40	60
16.5%	185	153	49	47	37	23	12	22	27	35	40	61
17.0%	169	128	50	46	34	23	11	21	25	34	37	60
17.5%	153	140	45	45	35	25	10	20	24	35	37	59
18.0%	138	145	44	44	33	24	9	19	25	33	35	59
18.5%	152	120	42	45	35	24	8	19	23	33	35	57
19.0%	130	113	52	42	31	22	4	17	22	33	36	58
19.5%	132	108	43	39	32	24	4	15	22	30	35	58
20.0%	133	90	40	46	31	23	3	16	21	30	34	58
20.5%	120	86	35	37	35	24	5	15	20	29	33	59
21.0%	113	85	38	40	29	22	5	13	21	29	33	59
21.5%	110	76	43	36	27	22	5	12	19	29	33	58
22.0%	102	73	36	36	28	23	6	12	19	28	34	59
22.5%	93	74	36	31	27	20	5	12	17	28	33	58
23.0%	86	77	34	33	26	22	6	11	18	27	32	59
23.5%	13	67	32	30	28	22	6	11	18	28	32	58
24.0%	24	67	31	34	24	22	6	11	16	28	31	59

Corporates, sovereigns, and banks; SMEs (5Mio. < Sales < 50 Mio.); SMEs (Sales < 5 Mio.); Mortgage; Revolving retail; Other retail

than for the case of deterministic LGDs. Even if the required portfolio size is still significantly smaller than with the ASRF solution (–81.50%), the accuracy is worse than for deterministic LGDs (+16.25%). This confirms the findings from before that the first-order adjustment is strictly preferable. The corresponding values for beta-distributed LGDs are almost identical (–81.50% and +16.38%).

4.3.4.5 Probing Granularity for Inhomogeneous Portfolios

Subsequently, the accuracy of the ES-based granularity adjustment will be tested for inhomogeneous portfolios, which consist of credits with different exposure weights and default probabilities. The high quality and low quality test portfolios are identical to those of Sect. 4.2.2.5. The analyzed portfolios consist of 40, 60, . . ., 400, 800, 1,600, and 4,000 loans and the Expected Shortfall is computed at a confidence level of 99.72% for a correlation parameter of $\rho = 20\%$. The resulting first- and second-order granularity add-on and the corresponding ES of a Monte Carlo simulation with three million trials are presented in Fig. 4.11.

The size and shape of the true and the approximated granularity add-ons are similar to those calculated for the VaR. Thus, we find that for the portfolio

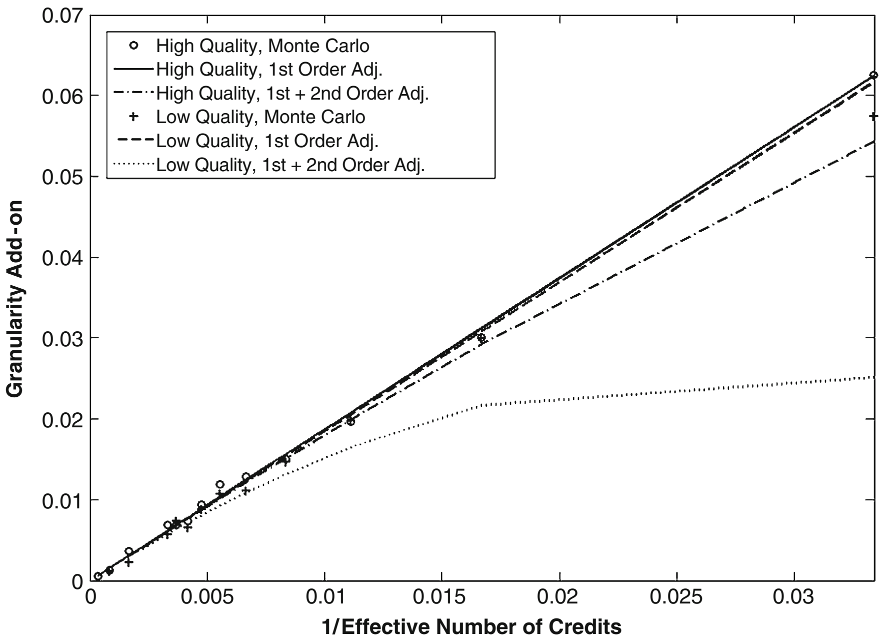

Fig. 4.11 ES-based granularity add-on for heterogeneous portfolios calculated analytically with first-order (*solid lines*) and second-order (*dotted lines*) adjustments as well as with Monte Carlo simulations (+ and o) using three million trials

consisting of 40 loans we have a granularity add-on of about 6%. In contrast to the VaR-based analysis, the add-on of the low-quality portfolio does not exceed the add-on of the high-quality portfolio. But most importantly, the granularity add-on is almost linear in terms of $1/n^*$ and the first-order adjustment is capable to capture the deviations from the ASRF solution with high accuracy, whereas the second-order adjustment leads to an underestimation of idiosyncratic risks.

4.4 Interim Result

Presently discussed analytical solutions for risk quantification of credit portfolio models especially rely on the assumptions of an infinite number of credits and of only one systematic factor. Thus, those analytical frameworks do not account for single name and sector concentration risk. This problem is discussed intensively by the financial authorities and it is especially considered in Pillar 2 of Basel II. To cope with the problem of name concentration, an add-on factor has been developed that adjusts the analytical solution for portfolios of finite size and therefore might serve as a simple solution for quantifying name concentration risk under Pillar 2. In this chapter, the general framework of this (first-order) granularity adjustment for medium sized risk buckets has been reviewed. Furthermore, we have derived an additional (second-order) adjustment for small risk buckets, which reduces the error term from $O(1/n^2)$ to $O(1/n^3)$. Even if it has already been mentioned by Gordy (2004) that it may be worthwhile to calculate these additional terms, the adjustment formula has not been determined before. After the derivation of the second-order-adjustment in general form, we have specified the formula for the Vasicek model. As a next step, we have carried out a detailed numerical study. In this study, we have reviewed the accuracy of the infinite granularity assumption for credit portfolios with a finite number of credits, as well as the improvement of accuracy with so-called first and second order granularity adjustments. Due to this study, banks are able to easily assess whether the assumption of infinite granularity is critical for their portfolio. Furthermore, the outcomes of the study show in which situations the granularity adjustment formulas are able to accurately measure portfolio name concentrations. These results are presented in terms of critical values for the minimum number of credits in a portfolio. We come to the conclusion that the critical number of credits for approving the assumption of infinite granularity is influenced by the probability of default, the asset correlation and of course the required accuracy of the analytical formula to great extent. We specify the minimum accuracy to 5%, i.e. if the credit portfolio is larger than our calculated critical values, the "true" risk and the approximation differ by less than 5%. This critical number of credits varies enormously, e.g. from 1,371 to 23,989 for a high-quality portfolio (A-rated) and from 23 to 205 for an extremely low-quality portfolio (CCC-rated) under the risk measure VaR. With the use of the first order granularity adjustment we can reduce these ranges drastically. The critical number of credits is in the bandwidth 456 to 4,227 (A-rated) and 9 to 42 (CCC-rated) and thus, the postulated accuracy should be obtained in many real-world portfolios. Additionally, the second order adjustment does not seem to work for the VaR since it reduces the add-on factor and the accuracy.

We have demonstrated that the VaR, which is coherent in the context of the ASRF framework, has some theoretical shortcomings if we leave the ASRF framework, which is necessary to account for name concentrations. For this reason, we have proposed a methodology how a more convenient risk measure can be used for the measurement of name concentrations. For this purpose, we have adjusted the confidence level of the ES in a way that the Pillar 1 formulas still lead to an almost identical capital requirement, leading to an ES-confidence level of $\alpha = 99.72\%$. Using this confidence level, we are able to measure name concentrations without being exposed to the theoretical shortcomings of the VaR, but the results are still consistent with the Pillar 1 formulas. Based on these preliminary considerations, we have theoretically derived the ES-based first- and second-order granularity adjustment in a general one-factor framework and for the Vasicek model. Similar to the corresponding formulas for the VaR, the second-order granularity adjustment, which is intended to improve the accuracy for small portfolios, has not been derived before in the literature. The subsequent numerical analyses confirm that the first-order granularity adjustment leads to a very good approximation of the unsystematic risk component whereas the second-order adjustment cannot improve the accuracy. Interestingly, the required portfolio size is not only 91.64% lower compared to the ASRF solution but also 49.05% lower compared to the VaR-based granularity adjustment. This shows that it is indeed advisable to measure name concentration risk on the basis of the coherent ES instead of relying on the non-coherent VaR.

These findings have been emphasized by a robustness check using stochastic LGDs. For this additional analysis, we have firstly calibrated several probability distributions with empirical data of recovery rates for different seniorities using a moment matching approach. Namely, we have used the normal distribution, the lognormal distribution, the logit-normal distribution, and the beta distribution. As the logit-normal distribution has performed best with respect to the empirical observed quantiles, we generated recovery rates which are logit-normal distributed with parameters stemming from the empirical data of senior unsecured loans. Using these data, we have repeated the test of the ASRF solution and the ES-based granularity adjustments. As expected, we find that the accuracy of the ASRF solution is lower due to the additional source of uncertainty. If the LGDs are stochastic, the minimum number of credits has to be 31.55% higher than for deterministic LGDs. Interestingly, the ES-based first-order adjustment performs slightly better in comparison with deterministic LGDs (4.89% less credits). Compared to the ASRF solution, the required portfolio size is 92.27% lower when using the first-order adjustment, which confirms our findings. Thus, apparently the accuracy of the measured risk is generally very high even for relatively small portfolios if the first-order granularity adjustment is incorporated.

4.5 Appendix

4.5.1 Alternative Derivation of the First-Order Granularity Adjustment

With reference to Wilde (2001), the granularity adjustment will be derived as an approximation of the difference Δq between the true VaR of a granular portfolio $q^{(n)}$ and the approximation $q^{(\infty)}$ that results if infinite granularity is assumed to hold:

$$\Delta q = q_\alpha^{(n)} - q_\alpha^{(\infty)}. \tag{4.104}$$

Instead of determining the add-on Δq directly, it will be analyzed how much the confidence level α will be overestimated or the probability $p := 1 - \alpha$ of exceeding the VaR will be underestimated if the portfolio is assumed to be infinitely granular. Thus, the probability

$$\Delta p = p^{(\infty)} - p = \alpha - \alpha^{(\infty)} \tag{4.105}$$

refers to the overestimation of the confidence level if only the systematic loss is considered. Here, α is the specified "target" confidence level, and by definition also the probability that the systematic loss will not exceed $q_\alpha^{(\infty)}$:

$$1 - p = \alpha := \mathbb{P}\left(\tilde{L} \le q_\alpha^{(n)}\right) = \mathbb{P}\left(\mathbb{E}\left[\tilde{L}\,|\,\tilde{x}\right] \le q_\alpha^{(\infty)}\right). \tag{4.106}$$

By contrast, $\alpha^{(\infty)}$ is the actual confidence level if the VaR is approximated by the ASRF model:

$$1 - p^{(\infty)} = \alpha^{(\infty)} := \mathbb{P}\left(\tilde{L} \le q_\alpha^{(\infty)}\right). \tag{4.107}$$

Subsequent to the derivation of Δp, the result will be transformed into a shift of the loss quantile Δq.

Analogous to Appendix 2.8.3, the unconditional probability $p^{(\infty)}$ can be expressed in terms of the conditional probability. Then, the substitution $y := q_\alpha^{(\infty)} + t$ is performed to center the integration at $q_\alpha^{(\infty)}$:

$$\begin{aligned} p + \Delta p = \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)}\right) &= \int\limits_{y=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)}\,|\,\tilde{Y} = y\right) f_Y(y)dy \\ &= \int\limits_{t=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \ge q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right)dt, \end{aligned} \tag{4.108}$$

with the shorter notation $\tilde{Y} := \mathbb{E}\left(\tilde{L}\,|\,\tilde{x}\right)$ for the conditional expectation. According to (4.106), the probability p can be written as

$$p = \mathbb{P}\left(\tilde{Y} \geq q_\alpha^{(\infty)}\right) = \int_{y=q_\alpha^{(\infty)}}^{\infty} f_Y(y)\,dy = \int_{t=0}^{\infty} f_Y\left(q_\alpha^{(\infty)} + t\right)dt \tag{4.109}$$

using the substitution $y := q_\alpha^{(\infty)} + t$ again, so that $t(y = q_\alpha^{(\infty)}) = 0$ and $t(y = \infty) = \infty$. Hence, (4.108) can be expressed as

$$\begin{aligned}
\Delta p &= \int_{t=-\infty}^{\infty} \mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right)dt - \int_{t=0}^{\infty} f_Y\left(q_\alpha^{(\infty)} + t\right)dt \\
&= \int_{t=-\infty}^{0} \mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) f_Y\left(q_\alpha^{(\infty)} + t\right)dt \\
&\quad + \int_{t=0}^{\infty} \left[\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) - 1\right] f_Y\left(q_\alpha^{(\infty)} + t\right)dt.
\end{aligned} \tag{4.110}$$

The following transformations are performed for simplification of the integrand in order to solve the integral. A realization of the systematic loss implies a realization of the systematic factor. As the credit loss events are assumed to be independent for a realization of the systematic factor, the conditional credit losses follow a binomial distribution, which can be approximated by a normal distribution for a sufficient number of credits. This leads to

$$\begin{aligned}
\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) &= 1 - \mathbb{P}\left(\tilde{L} < q_\alpha^{(\infty)}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) \\
&\approx 1 - \Phi\left(\frac{q_\alpha^{(\infty)} - \mathbb{E}\left(\tilde{L}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right)}{\sqrt{\mathbb{V}\left(\tilde{L}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right)}}\right).
\end{aligned} \tag{4.111}$$

As $\mathbb{E}(\tilde{L}) = \mathbb{E}(\mathbb{E}(\tilde{L}\,|\,\tilde{x})) = \mathbb{E}(\tilde{Y})$, which is due to the law of iterated expectation, the conditional expectation of (4.111) equals

$$\mathbb{E}\left(\tilde{L}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) = \mathbb{E}\left(\tilde{Y}\,|\,\tilde{Y} = q_\alpha^{(\infty)} + t\right) = q_\alpha^{(\infty)} + t. \tag{4.112}$$

With the symmetry $1 - \Phi(-x) = \Phi(x)$ and defining $\sigma^2(y) := \mathbb{V}(\tilde{L}\,|\,\tilde{Y} = y)$, (4.111) results in

$$\mathbb{P}\left(\tilde{L} \geq q_\alpha^{(\infty)} \mid \tilde{Y} = q_\alpha^{(\infty)} + t\right) \approx 1 - \Phi\left(\frac{q_\alpha^{(\infty)} - q_\alpha^{(\infty)} - t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right)$$
$$= \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right), \tag{4.113}$$

so that (4.110) can be written as

$$\Delta p = \int_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) f_Y\left(q_\alpha^{(\infty)} + t\right) dt$$
$$+ \int_{t=0}^{\infty} \left[\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) - 1\right] f_Y\left(q_\alpha^{(\infty)} + t\right) dt. \tag{4.114}$$

Subsequently, several linear approximations will be performed relying on the assumption that the loss quantile of the granular portfolio is close to the systematic loss quantile and the linearizations lead to minor errors. Linearizing the density function at $q_\alpha^{(\infty)}$ leads to

$$f_Y\left(q_\alpha^{(\infty)} + t\right) \approx f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \left.\frac{df_Y(y)}{dy}\right|_{y=q_\alpha^{(\infty)}}. \tag{4.115}$$

The argument of the normal distribution can be approximated as

$$t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right) \approx t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)}\right)} + t \cdot \left[\frac{d}{dt} \frac{1}{\sigma\left(q_\alpha^{(\infty)} + t\right)}\right]_{t=0}\right)$$
$$= t \cdot \left(\frac{1}{\sigma\left(q_\alpha^{(\infty)}\right)} + t \cdot \left[-\frac{1}{\sigma^2\left(q_\alpha^{(\infty)} + t\right)} \frac{d}{dt} \sigma\left(q_\alpha^{(\infty)} + t\right)\right]_{t=0}\right)$$
$$= \left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \left[\frac{d}{dt} \sigma\left(q_\alpha^{(\infty)} + t\right)\right]_{t=0}\right). \tag{4.116}$$

With the substitution $y := q_\alpha^{(\infty)} + t$, so $dy/dt = 1$ and $y(t=0) = q_\alpha^{(\infty)}$, the derivative of the conditional standard deviation can be rewritten as

$$\left.\frac{d}{dt} \sigma\left(q_\alpha^{(\infty)} + t\right)\right|_{t=0} = \frac{d}{dy} \sigma(y)\big|_{y=q_\alpha^{(\infty)}}. \tag{4.117}$$

Inserting (4.115)–(4.117) in (4.114) leads to

$$\begin{aligned}\Delta p = &\left(\int\limits_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) \right. \\ &\left. \cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] dt \right) \\ &- \left(- \int\limits_{t=0}^{\infty} \left[\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} - \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) - 1 \right] \right. \\ &\left. \cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) + t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] dt \right) \\ &=: \Delta p_1 - \Delta p_2. \end{aligned} \tag{4.118}$$

When the substitution $t := -t$ for the term Δp_2 is performed and the symmetry of the normal distribution $\Phi(-x) - 1 = -\Phi(x)$ is used, both terms Δp_1 and Δp_2 are identical except for the algebraic signs:

$$\begin{aligned}\Delta p_2 = &- \int\limits_{t=0}^{-\infty} \left[\Phi\left(- \left[\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} + \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] \right) - 1 \right] \\ &\cdot \left[f_Y\left(q_\alpha^{(\infty)}\right) - t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right] \cdot (-1) dt \\ = &\int\limits_{t=-\infty}^{0} \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} + \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) \\ &\cdot \left(f_Y\left(q_\alpha^{(\infty)}\right) - t \cdot \frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) dt. \end{aligned} \tag{4.119}$$

A linearization of the normal distributions in Δp_1 and Δp_2 results in

$$\begin{aligned}&\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \right) \\ &\quad \approx \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right) \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \frac{d\Phi(y)}{dy}\bigg|_{y=\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}} \\ &\quad = \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right) \mp \frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)} \right). \end{aligned} \tag{4.120}$$

Using this approximation, the terms Δp_1 and Δp_2 from (4.118) can be written as

$$\Delta p_{1,2} \approx \int\limits_{t=-\infty}^{0} \underbrace{\Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right)}_{=:\beta_0} \cdot \left[\underbrace{f_Y\left(q_\alpha^{(\infty)}\right)}_{=:\gamma_0} \pm \underbrace{t\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}}_{=:\gamma_1}\right] dt$$
$$\mp \int\limits_{t=-\infty}^{0} \underbrace{\frac{t^2}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right)}_{=:\beta_1} \cdot \left[\underbrace{f_Y\left(q_\alpha^{(\infty)}\right)}_{=:\gamma_0} \pm \underbrace{t\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}}_{=:\gamma_1}\right] dt. \tag{4.121}$$

The summands $\beta_0, \ \gamma_0$ are the points around which the linearizations have been performed. The summands $\beta_1, \ \gamma_1$ have resulted from the first-order approximations. Using this notation, the shift in probability Δp of (4.118) can notably be simplified to

$$\begin{aligned} \Delta p &\approx \Delta p_1 - \Delta p_2 \\ &\approx \int\limits_{t=-\infty}^{0} \beta_0(\gamma_0 + \gamma_1) - \beta_1(\gamma_0 + \gamma_1) dt - \int\limits_{t=-\infty}^{0} \beta_0(\gamma_0 - \gamma_1) + \beta_1(\gamma_0 - \gamma_1) dt \\ &= \int\limits_{t=-\infty}^{0} 2\beta_0\gamma_1 - 2\beta_1\gamma_0 dt. \end{aligned} \tag{4.122}$$

Fortunately, both integrands are already first-order terms whereas the cross-terms $\beta_1 \cdot \gamma_1$ vanish.[232] Thus, there is no need for a further linearization. The remaining expression is

$$\begin{aligned} \Delta p \approx\ & 2\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \int\limits_{t=-\infty}^{0} t \cdot \Phi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right) dt \\ & - 2\frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} \frac{f_Y\left(q_\alpha^{(\infty)}\right)}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \int\limits_{t=-\infty}^{0} t^2 \cdot \varphi\left(\frac{t}{\sigma\left(q_\alpha^{(\infty)}\right)}\right) dt. \end{aligned} \tag{4.123}$$

[232] The omission of the zeroth-order terms could be foreseen as only the *deviation* from the systematic loss quantile is analyzed.

In order to solve the integrals, the substitution $y := t/\sigma(q_\alpha^{(\infty)})$ is performed, with $dy/dt = 1/\sigma(q_\alpha^{(\infty)})$, $y(t=-\infty) = -\infty$ and $y(t=0) = 0$:

$$\begin{aligned}\Delta p \approx\ & 2\left.\frac{df_Y(y)}{dy}\right|_{y=q_\alpha^{(\infty)}} \int\limits_{y=-\infty}^{0} y\cdot\sigma\left(q_\alpha^{(\infty)}\right)\cdot\Phi(y)\cdot\sigma\left(q_\alpha^{(\infty)}\right)dy \\ & -2\left.\frac{d\sigma(y)}{dy}\right|_{y=q_\alpha^{(\infty)}} \frac{f_Y\left(q_\alpha^{(\infty)}\right)}{\sigma^2\left(q_\alpha^{(\infty)}\right)} \int\limits_{y=-\infty}^{0}\left[y\cdot\sigma\left(q_\alpha^{(\infty)}\right)\right]^2\cdot\varphi(y)\cdot\sigma\left(q_\alpha^{(\infty)}\right)dy \\ =\ & 2\left.\frac{df_Y(y)}{dy}\right|_{y=q_\alpha^{(\infty)}} \sigma^2\left(q_\alpha^{(\infty)}\right) \underbrace{\int\limits_{y=-\infty}^{0} y\cdot\Phi(y)dy}_{*} \\ & -2\left.\frac{d\sigma(y)}{dy}\right|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\cdot\sigma\left(q_\alpha^{(\infty)}\right) \underbrace{\int\limits_{y=-\infty}^{0} y^2\cdot\varphi(y)dy}_{**}.\end{aligned} \tag{4.124}$$

For the second integral (**), it is used that the integrand is axially symmetric to the y-axis. Furthermore, the definition of the variance is utilized, considering that the standard normal distribution has mean $\mu_Y = 0$ and variance $\sigma_Y^2 = 1$:

$$\begin{aligned}\int\limits_{y=-\infty}^{0} y^2\cdot\varphi(y)dy &= \frac{1}{2}\int\limits_{y=-\infty}^{\infty} y^2\cdot\varphi(y)dy = \frac{1}{2}\int\limits_{y=-\infty}^{\infty}(y-\mu_Y)^2\cdot\varphi(y)dy. \\ &= \frac{1}{2}{\sigma_Y}^2 = \frac{1}{2}.\end{aligned} \tag{4.125}$$

The first integral (*) can be calculated with integration by parts:

$$\int\limits_{y=-\infty}^{0} y\cdot\Phi(y)dy = \left[\frac{1}{2}y^2\cdot\Phi(y)\right]_{y=-\infty}^{0} - \int\limits_{y=-\infty}^{0}\frac{1}{2}y^2\cdot\varphi(y)\,dy. \tag{4.126}$$

For $y = 0$, the first term is zero but for $y = -\infty$, the result is not obvious. Using l'Hôpital's rule several times leads to[233]

[233]For functions f, g with $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ or $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$ it is true that $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)} = \lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$ if $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}$ exists; cf. Bronshtein et al. (2007), p. 54, (2.26).

$$
\begin{aligned}
\lim_{y\to-\infty}\frac{1}{2}y^2\cdot\Phi(y) &= \lim_{y\to\infty}\frac{1}{2}\frac{\Phi(-y)}{y^{-2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{1}{2}\frac{-\varphi(-y)}{-2y^{-3}} \\
&= \lim_{y\to\infty}\frac{1}{4}\frac{y^3}{e^{y^2/2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{1}{4}\frac{3y^2}{y\cdot e^{y^2/2}} \\
&= \lim_{y\to\infty}\frac{3}{4}\frac{y}{e^{y^2/2}} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty}\frac{3}{4}\frac{1}{y\cdot e^{y^2/2}} = 0,
\end{aligned} \tag{4.127}
$$

so that the first term of (4.126) vanishes. Using the result of the previous integration, (4.126) equals $-1/4$. Hence, Δp from (4.124) is given as

$$
\Delta p \approx -\frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\sigma^2\left(q_\alpha^{(\infty)}\right) - \frac{d\sigma(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\cdot\sigma\left(q_\alpha^{(\infty)}\right). \tag{4.128}
$$

Because of $\sigma\frac{d\sigma}{dy} = \frac{1}{2}\frac{d\sigma^2}{d\sigma}\frac{d\sigma}{dy} = \frac{1}{2}\frac{d\sigma^2}{dy}$, (4.128) is equivalent to

$$
\begin{aligned}
\Delta p &\approx -\left[\frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\sigma^2\left(q_\alpha^{(\infty)}\right) + \frac{1}{2}\frac{d\sigma^2(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}} f_Y\left(q_\alpha^{(\infty)}\right)\right] \\
&= -\frac{1}{2}\left[\frac{df_Y(y)}{dy}\sigma^2(y) + \frac{d\sigma^2(y)}{dy}f_Y(y)\right]_{y=q_\alpha^{(\infty)}} \\
&= -\frac{1}{2}\frac{d}{dy}\left(f_Y(y)\cdot\sigma^2(y)\right)\Big|_{y=q_\alpha^{(\infty)}}.
\end{aligned} \tag{4.129}
$$

This expression is the linearized deviation of the specified probability $p = 1-\alpha$ if only the systematic loss is considered for calculation of the loss quantile.

As initially noticed, the determined shift of the probability has to be transformed into a shift of the loss quantile (cf. Fig. 4.12). If the probability density function of the portfolio loss is assumed to be almost linear in a region around the quantile, the required transformation is

$$
\Delta p \approx \frac{1}{2}\left[f_Y\left(q_\alpha^{(\infty)}\right) + f_Y\left(q_\alpha^{(\infty)} + \Delta q\right)\right]\Delta q. \tag{4.130}
$$

Two last first-order approximations lead to

$$
\begin{aligned}
\Delta p &\approx \frac{1}{2}\left[f_Y\left(q_\alpha^{(\infty)}\right) + \left(f_Y\left(q_\alpha^{(\infty)}\right) + \Delta q\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}\right)\right]\Delta q \\
&= f_Y\left(q_\alpha^{(\infty)}\right)\cdot\Delta q + \frac{1}{2}\frac{df_Y(y)}{dy}\bigg|_{y=q_\alpha^{(\infty)}}(\Delta q)^2 \\
&\approx f_Y\left(q_\alpha^{(\infty)}\right)\cdot\Delta q.
\end{aligned} \tag{4.131}
$$

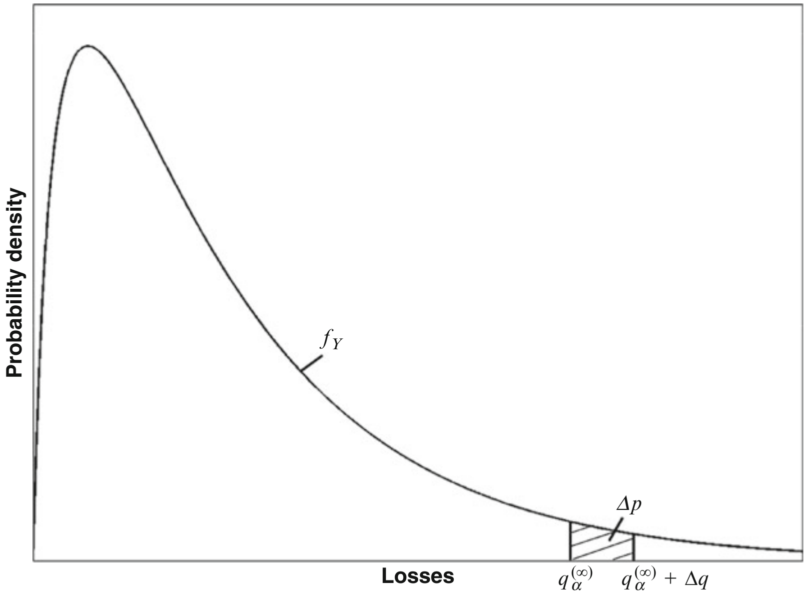

Fig. 4.12 Relation between the shift of the probability and the loss quantile

Inserting (4.129) into (4.131) finally leads to

$$\Delta q \approx \frac{\Delta p}{f_Y\left(q_\alpha^{(\infty)}\right)} \approx -\frac{1}{2}\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\cdot\sigma^2(y)\right)\Big|_{y=q_\alpha^{(\infty)}}$$
$$= -\frac{1}{2}\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\cdot\mathbb{V}\left(\tilde{L}\,|\,\tilde{Y}=y\right)\right)\Big|_{y=q_\alpha^{(\infty)}}. \quad (4.132)$$

Using (4.8), this can be written as

$$\Delta q \approx -\frac{1}{2f_x(x)}\frac{d}{dx}\left(\frac{f_x(x)\mathbb{V}\left[\tilde{L}\,|\,\tilde{x}=x\right]}{\frac{d}{dx}\mathbb{E}\left[\tilde{L}\,|\,\tilde{x}=x\right]}\right)\Bigg|_{x=q_{1-\alpha}(\tilde{x})}, \quad (4.133)$$

which is identical to the first-order granularity adjustment of Sect. 4.2.1.1.[234]

4.5.2 First and Second Derivative of VaR

The derivatives of VaR will be determined on the basis of Rau-Bredow (2002, 2004) in the following. Consider two continuous random variables $\tilde{Y}$ and $\tilde{Z}$ with

[234]Cf. Wilde (2001).

joint probability density function $f(y,z)$ and a variable $\lambda \in \mathbb{R}$. The VaR (the quantile) $q := q_\alpha(\tilde{L})$ of $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ can implicitly be defined as[235]

$$\mathbb{P}(\tilde{L} \leq q) = \alpha. \tag{4.134}$$

Furthermore, the formula of the conditional density function will be used:[236]

$$f_{Z\,|\,Y=y}(z) = \frac{f_{Y,\,Z}(y,z)}{f_Y(y)}, \tag{4.135}$$

leading to[237]

$$f_{Z\,|\,Y+\lambda Z=q}(z) = \frac{f_{Y+\lambda Z,Z}(q,z)}{f_{Y+\lambda Z}(q)} = \frac{f_{Y,\,Z}(q-\lambda z, z)}{f_{Y+\lambda Z}(q)}. \tag{4.136}$$

4.5.2.1 First Derivative

As the derivative of the constant α is zero, the derivative of (4.134) is

$$\begin{aligned} 0 &= \frac{\partial}{\partial\lambda}\mathbb{P}(\tilde{Y} + \lambda\tilde{Z} \leq q) \\ &= \frac{\partial}{\partial\lambda} \int_{z=-\infty}^{\infty} \int_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\,dy\,dz \\ &= \int_{z=-\infty}^{\infty} \frac{\partial}{\partial\lambda} \int_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\,dy\,dz. \end{aligned} \tag{4.137}$$

Performing the inner integration and the differentiation leads to

$$0 = \int_{z=-\infty}^{\infty} \left(\frac{dq}{d\lambda} - z\right) f_{Y,\,Z}(q-\lambda z, z)\,dz. \tag{4.138}$$

[235]Cf. (2.14). The slightly different expressions compared to Rau-Bredow (2002) result from α instead of $(1–\alpha)$ representing the confidence level.

[236]Cf. Pitman (1999), p. 416.

[237]Cf. Rau-Bredow (2004), p. 66.

Using the formula for the conditional density function (4.135) and the integral representation of the conditional expectation, we get

$$\begin{aligned}
0 &= \int_{z=-\infty}^{\infty} \left(\frac{dq}{d\lambda} - z\right) f_{Y+\lambda Z}(q) f_{Z\,|\,Y+\lambda Z=q}(z)\, dz \\
&= f_{Y+\lambda Z}(q)\left(\frac{dq}{d\lambda}\int_{z=-\infty}^{\infty} f_{Z\,|\,Y+\lambda Z=q}(z)\, dz - \int_{z=-\infty}^{\infty} z f_{Z\,|\,Y+\lambda Z=q}(z)\, dz\right) \\
&= f_{Y+\lambda Z}(q)\left(\frac{dq}{d\lambda}\cdot 1 - \mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=q\right]\right).
\end{aligned} \tag{4.139}$$

This leads to the first derivative of VaR:

$$\frac{dVaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda} = \mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z} = q_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)\right]. \tag{4.140}$$

The first derivative at $\lambda = 0$ is

$$\left.\frac{dVaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda}\right|_{\lambda=0} = \mathbb{E}\left[\tilde{Z}\,|\,\tilde{Y} = q_\alpha\left(\tilde{Y}\right)\right]. \tag{4.141}$$

4.5.2.2 Second Derivative

Similar to (4.137), the second derivative of (4.134) is

$$0 = \frac{\partial^2}{\partial\lambda^2}\mathbb{P}\left(\tilde{Y}+\lambda\tilde{Z}\le q\right) = \frac{\partial^2}{\partial\lambda^2}\int_{z=-\infty}^{\infty}\int_{y=-\infty}^{q-\lambda z} f_{Y,\,Z}(y,z)\, dy\, dz. \tag{4.142}$$

With the first derivative of (4.138) and applying the product rule, this leads to

$$\begin{aligned}
0 &= \frac{\partial}{\partial\lambda}\int_{z=-\infty}^{\infty}\left(\frac{dq}{d\lambda} - z\right) f_{Y,\,Z}(q-\lambda z, z)\, dz \\
&= \int_{z=-\infty}^{\infty}\left(\frac{d^2 q}{d^2\lambda}\right) f_{Y,\,Z}(q-\lambda z, z) + \left(\frac{dq}{d\lambda} - z\right)\underbrace{\frac{\partial f_{Y,\,Z}(q-\lambda z, z)}{\partial\lambda}}_{*} dz.
\end{aligned} \tag{4.143}$$

The derivative (*) can be determined with the chain rule:

$$
\begin{aligned}
\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial \lambda} &= \frac{\partial(q-\lambda z)}{\partial \lambda}\,\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial(q-\lambda z)}\,\frac{\partial q}{\partial q}\\
&= \left(\frac{dq}{d\lambda}-z\right)\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}\,\frac{1}{\partial(q-\lambda z)/\partial q}\\
&= \left(\frac{dq}{d\lambda}-z\right)\frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}\,.
\end{aligned}
\tag{4.144}
$$

Inserting (4.144) and the conditional density (4.136) into (4.143) results in

$$
\begin{aligned}
0 &= \int\limits_{z=-\infty}^{\infty} \left(\frac{d^2q}{d^2\lambda}\right) f_{Y,Z}(q-\lambda z,z) + \left(\frac{dq}{d\lambda}-z\right)^2 \frac{\partial f_{Y,Z}(q-\lambda z,z)}{\partial q}\,dz\\
&= \left(\frac{d^2q}{d^2\lambda}\right) \int\limits_{z=-\infty}^{\infty} f_{Y+\lambda Z}(q)\, f_{Z\,|\,Y+\lambda Z=q}(z)dz\\
&\quad + \int\limits_{z=-\infty}^{\infty} \left(\frac{dq}{d\lambda}-z\right)^2 \frac{\partial\big(f_{Y+\lambda Z}(q) f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}\,dz.
\end{aligned}
\tag{4.145}
$$

The first summand of (4.145) equals

$$
\left(\frac{d^2q}{d^2\lambda}\right) f_{Y+\lambda Z}(q) \int\limits_{z=-\infty}^{\infty} f_{Z\,|\,Y+\lambda Z=q}(z)dz = \left(\frac{d^2q}{d^2\lambda}\right) f_{Y+\lambda Z}(q). \tag{4.146}
$$

In order to calculate the second summand of (4.145), the first derivative from (4.140) as well as the integral representation of the conditional variance is used:

$$
\begin{aligned}
&\int\limits_{z=-\infty}^{\infty} \left(\frac{dq}{d\lambda}-z\right)^2 \frac{\partial\big(f_{Y+\lambda Z}(q) f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}\,dz\\
&= \int\limits_{z=-\infty}^{\infty} \big(z-\mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=q\big]\big)^2 \frac{\partial\big(f_{Y+\lambda Z}(q) f_{Z\,|\,Y+\lambda Z=q}(z)\big)}{\partial q}\,dz\\
&= \frac{d}{dy}\left(f_{Y+\lambda Z}(y) \int\limits_{z=-\infty}^{\infty} \big(z-\mathbb{E}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=q\big]\big)^2 f_{Z\,|\,Y+\lambda Z=y}(z)dz\right)\Bigg|_{y=q}\\
&= \frac{d}{dy}\big(f_{Y+\lambda Z}(y)\mathbb{V}\big[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=y\big]\big)\Big|_{y=q}.
\end{aligned}
\tag{4.147}
$$

With these summands, (4.145) can be written as

$$0 = \left(\frac{d^2q}{d^2\lambda}\right) f_{Y+\lambda Z}(y) + \frac{d}{dy}\left(f_{Y+\lambda Z}(y)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=y\right]\right)\Big|_{y=q}. \tag{4.148}$$

Thus, the second derivative of VaR is equal to

$$\frac{d^2VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda} = -\frac{1}{f_{Y+\lambda Z}(y)}\cdot\frac{d}{dy}\left(f_{Y+\lambda Z}(y)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}+\lambda\tilde{Z}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}. \tag{4.149}$$

The second derivative at $\lambda = 0$ is

$$\frac{d^2VaR_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d^2\lambda}\Bigg|_{\lambda=0} = -\frac{1}{f_Y(y)}\frac{d}{dy}\left(f_Y(y)\mathbb{V}\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)\Big|_{y=q_\alpha\left(\tilde{Y}\right)}. \tag{4.150}$$

4.5.3 Probability Density Function of Transformed Random Variables

Let $\tilde{X}$ be a random variable with density $f_X(x)$ and let $\tilde{Y}$ be a random variable with $\tilde{Y} = g(\tilde{X})$. If g is strictly monotonous and differentiable, the probability density function (PDF) of $\tilde{Y}$ can be transformed using the inverse function theorem[238]:

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\cdot\left|\frac{dg^{-1}(y)}{dy}\right|. \tag{4.151}$$

With $g^{-1}(y) = x$, we obtain

$$\left|\frac{dg^{-1}(y)}{dy}\right| = \left|\frac{dx}{dy}\right| = \left|\frac{1}{dy/dx}\right|, \tag{4.152}$$

which leads to

$$f_Y(y) = \frac{f_X(x)}{|dy/dx|}. \tag{4.153}$$

[238]Cf. Roussas (2007), p. 236.

4.5.4 VaR-Based First-Order Granularity Adjustment for a Normally Distributed Systematic Factor

The granularity adjustment (4.10) can be expressed as

$$\begin{aligned}\Delta l_1 &= -\frac{1}{2\varphi}\frac{d}{dx}\left(\frac{\varphi\,\eta_{2,c}}{d\mu_{1,c}/dx}\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)}\\
&= -\frac{1}{2\varphi}\left[\frac{d}{dx}\left(\varphi\,\eta_{2,c}\right)\frac{1}{d\mu_{1,c}/dx}+\varphi\,\eta_{2,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}\\
&= -\frac{1}{2}\left[\frac{1}{\varphi}\frac{d}{dx}\left(\varphi\,\eta_{2,c}\right)\frac{1}{d\mu_{1,c}/dx}+\eta_{2,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}\\
&= -\frac{1}{2}\left[\left(\frac{\eta_{2,c}}{\varphi}\frac{d\varphi}{dx}+\frac{d\eta_{2,c}}{dx}\right)\frac{1}{d\mu_{1,c}/dx}-\eta_{2,c}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}.\end{aligned} \tag{4.154}$$

Because of

$$\frac{1}{\varphi}\frac{d\varphi}{dx}=\frac{d(\ln\varphi)}{dx}=\frac{d}{dx}\left(\ln\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\right]\right)=\frac{d}{dx}\left(\ln\frac{1}{\sqrt{2\pi}}-\frac{x^2}{2}\right)=-x, \tag{4.155}$$

the granularity adjustment (4.154) can be written as

$$\Delta l_1 = \frac{1}{2}\left[\frac{x\cdot\eta_{2,c}}{d\mu_{1,c}/dx}-\frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx}+\frac{\eta_{2,c}\cdot d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \tag{4.156}$$

For the calculation of (4.156), the conditional expectation and variance have to be determined. Assuming stochastically independent LGDs and with *ELGD* and *VLGD* for the expectation and the variance of the LGD, respectively, the required moments are given as[239]

$$\begin{aligned}\mu_{1,c} &= \mathbb{E}\left(\sum_{i=1}^{n} w_i\cdot\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right)\\
&= \sum_{i=1}^{n} w_i\cdot ELGD_i\cdot\mathbb{E}\left(1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right)\\
&= \sum_{i=1}^{n} w_i\cdot ELGD_i\cdot p_i(x),\end{aligned} \tag{4.157}$$

[239] Pykhtin and Dev (2002) corrected the formulas of Wilde (2001), who neglected the last term of the following conditional variance.

$$\begin{aligned}
\eta_{2,c} &= \mathbb{V}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \mid \tilde{x} = x\right) \\
&= \sum_{i=1}^{n} w_i^2 \cdot \mathbb{V}\Big(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \mid \tilde{x} = x\Big) \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\mathbb{E}\left(\left[\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \mid \tilde{x} = x\right]^2\right) - \mathbb{E}^2\Big(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \mid \tilde{x} = x\Big)\right] \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\mathbb{E}\Big(\widetilde{LGD}_i^{\,2}\Big) \cdot \mathbb{E}\left(\left[1_{\{\tilde{D}_i\}} \mid \tilde{x} = x\right]^2\right) - (ELGD_i \cdot p_i(x))^2\right] \\
&= \sum_{i=1}^{n} w_i^2 \cdot \left[\left(ELGD_i^2 + VLGD_i\right) \cdot p_i(x) - ELGD_i^2 \cdot p_i^2(x)\right].
\end{aligned} \tag{4.158}$$

4.5.5 VaR-Based First-Order Granularity Adjustment for Homogeneous Portfolios

For homogeneous portfolios, the granularity adjustment formula (4.28) can be simplified to

$$\begin{aligned}
\Delta l_1 &= \frac{1}{2n}\Bigg[\Phi^{-1}(\alpha)\frac{(ELGD^2 + VLGD)\Phi(z) - ELGD^2\,\Phi^2(z)}{ELGD\big(\sqrt{\rho}/\sqrt{1-\rho}\big)\varphi(z)} \\
&\quad - \frac{(ELGD^2 + VLGD) - 2ELGD^2\,\Phi(z)}{ELGD} \\
&\quad - \frac{(ELGD^2 + VLGD)\Phi(z)z - ELGD^2\,\Phi^2(z)z}{ELGD \cdot \varphi(z)}\Bigg]_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}} \\
&= \frac{1}{2n}\Bigg(\frac{ELGD^2 + VLGD}{ELGD}\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(\alpha)\Phi(z)}{\sqrt{\rho}\,\varphi(z)} - 1 - \frac{\Phi(z)z}{\varphi(z)}\right] \\
&\quad - ELGD\,\Phi(z)\left[\frac{\sqrt{1-\rho}\,\Phi^{-1}(\alpha)\Phi(z)}{\sqrt{\rho}\,\varphi(z)} - 2 - \frac{\Phi(z)z}{\varphi(z)}\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}} \\
&= \frac{1}{2n}\Bigg(\frac{ELGD^2 + VLGD}{ELGD}\left[\frac{\Phi(z)}{\varphi(z)}\,\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 1\right] \\
&\quad - ELGD \cdot \Phi(z)\left[\frac{\Phi(z)}{\varphi(z)}\,\frac{\Phi^{-1}(\alpha)(1-2\rho) + \Phi^{-1}(PD)\sqrt{\rho}}{\sqrt{\rho}\sqrt{1-\rho}} - 2\right]\Bigg)_{z=\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}}.
\end{aligned} \tag{4.159}$$

4.5.6 Arbitrary Derivatives of VaR

The following determination of all derivatives of VaR is based on Wilde (2003). The quantile q_α of $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ can be written as $q(\lambda)$ to denote that the quantile depends on the parameter λ. Using this notation, the quantile can be defined implicitly as an argument of the distribution function F by $F(q(\lambda), \lambda) := \mathbb{P}\big(\tilde{Y} + \lambda\tilde{Z} \leq q_\alpha(\tilde{Y} + \lambda\tilde{Z})\big) = \alpha$. In order to calculate the derivatives of q_α, at first all derivatives of F are determined in Sect. 4.5.6.2.1. As the quantile is defined implicitly, the implicit derivatives of $F(q(\lambda), \lambda) - \alpha = 0$ have to be determined. This is done by application of the residue theorem in Sect. 4.5.6.2.2. As a next step, the result will be expressed in combinatorial form in Sect. 4.5.6.2.3. Using the results of the derivatives of the distribution function and the implicit derivatives, it is possible to determine all derivatives of VaR. This is performed in Sect. 4.5.6.2.4. As the resulting formula is quite complex, an expression for the first five derivatives of VaR is determined in Sect. 4.5.7. The mathematical basics to the Laplace transform, complex residues, and partitions, which are needed within the derivation, are presented in the following Sect. 4.5.6.1.

4.5.6.1 Mathematical Basics

4.5.6.1.1 Laplace Transform and Dirac's Delta Function

The *Laplace transform* $\mathcal{L}$ of a function $f(t)$ with $t \in \mathbb{R}^+$ is given as[240]

$$[\mathcal{L}\{f(t)\}](s) := \int_{t=-0}^{\infty} f(t)e^{-st}dt =: \Theta(s) \tag{4.160}$$

with $s = c + i\omega \in \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers. The *inverse Laplace transform* $\mathcal{L}^{-1}$ can be represented as[241]

$$\left[\mathcal{L}^{-1}\{\Theta(s)\}\right](t) := \frac{1}{2\pi i}\int_{s=c-i\infty}^{c+i\infty} \Theta(s)\mathrm{e}^{st}ds = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t). \tag{4.161}$$

Dirac's delta function $\delta(x)$ can be defined as[242]

$$\int_{-\infty}^{\infty} \delta(x)f(x-x_0)dx = f(x_0). \tag{4.162}$$

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[240] Cf. Bronshtein et al. (2007), p. 710, (15.5).

[241] Cf. Bronshtein et al. (2007), p. 710, (15.8).

[242] Weisstein (2009a).

A more illustrative, heuristic definition of $\delta(x)$ is given by

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases} \quad \text{and} \quad \int\limits_{-\infty}^{\infty} \delta(x)dx = 1. \tag{4.163}$$

Using the definition of the Laplace transform and the inverse Laplace transform, Dirac's delta function can be written as

$$\begin{aligned} \delta(t) &= \mathcal{L}^{-1}\{\mathcal{L}\{\delta(t)\}\} = \mathcal{L}^{-1}\left\{ \int\limits_{t=-0}^{\infty} \delta(t)\mathrm{e}^{-st}dt \right\} \\ &= \mathcal{L}^{-1}\left\{\mathrm{e}^{-s\cdot 0}\right\} = \mathcal{L}^{-1}\{1\} = \frac{1}{2\pi i} \int\limits_{s=c-i\infty}^{c+i\infty} 1 \cdot \mathrm{e}^{st}ds. \end{aligned} \tag{4.164}$$

4.5.6.1.2 Laurent Series, Singularities, and Complex Residues

If $f(z)$ is differentiable in all points of an open subset of the complex plane $H \subset \mathbb{C}$, then we call $f(z)$ *holomorphic* on H.[243] For a function $f(z)$, which is holomorphic in a simply connected region H, according to the *Cauchy integral theorem* we have[244]

$$\oint\limits_C f(z)dz = 0, \tag{4.165}$$

with C being a closed path in H. If a function $f(z)$ is holomorphic in z_0 and in a circular region around z_0, we can perform a *Taylor series expansion*, which is analogous to the real plane:[245]

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{4.166}$$

However, if a function $f(z)$ is only holomorphic inside the annulus between two concentric circles with center z_0 and radii r_1 and r_2, which is the region

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[243] Cf. Bronshtein et al. (2007), p. 672, Sect. 14.1.2.1.

[244] Cf. Bronshtein et al. (2007), p. 688, (14.41).

[245] Cf. Bronshtein et al. (2007), p. 691, (14.49).

$H = \{z \mid 0 \leq r_1 < |z - z_0| < r_2\}$, the function $f(z)$ can be expressed as a generalized power series, the so-called *Laurent series*:[246]

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \underbrace{\sum_{n=-\infty}^{-1} a_n (z - z_0)^n}_{\text{principal part}} + \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{analytic part}}. \tag{4.167}$$

Thus, the function has to be holomorphic only inside the annulus and not inside the inner circle or outside the outer circle.

If a function $f(z)$ is holomorphic in a neighborhood of z_0 but not in the point z_0, then z_0 is called an *isolated singularity* of the function $f(z)$. The concrete type of a singularity can be classified according to the analytic part of the Laurent series:[247]

- The point z_0 is a *removable singularity* if $a_n = 0 \; \forall n < 0$. In this case, the Laurent series is identical to the Taylor series above.
- The point z_0 is a *pole of order m* if the principal part consists of a finite number of terms with $a_m \neq 0$ and $a_n = 0$ for $n < m < 0$.
- The point z_0 is an *essential singularity* if the principal part consists of an infinite number of terms.

The coefficient a_{-1} of the Laurent series (4.167) around an isolated singularity z_0 is the *residue* of $f(z)$ in z_0. This will subsequently be denoted by $\text{Res}_{z_0}(f)$. The residue can also be defined as

$$a_{-1} = \text{Res}_{z_0}(f) = \frac{1}{2\pi i} \cdot \oint_C f(z)\, dz, \tag{4.168}$$

where C is a contour with winding number 1 in a holomorphic region H around an isolated singularity in z_0. If the contour C encloses a finite number of isolated singularities $z_1, z_2, ..., z_m$ with corresponding residues $a_{-1}(z_\mu)$ $(\mu = 1, ..., m)$, we have

$$\oint_C f(z) dz = 2\pi i \sum_{\mu=1}^{m} a_{-1}(z_\mu), \tag{4.169}$$

which is the *residue theorem*.[248]

The residue $\text{Res}_{z_0}(f)$ with z_0 being a pole of order m can be calculated as[249]

$$\text{Res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m \cdot f(z)]. \tag{4.170}$$

[246] Cf. Bronshtein et al. (2007), p. 692, (14.51), and Spiegel (1999), p. 144.

[247] Cf. Bronshtein et al. (2007), p. 692 f., Sect. 14.3.5.1.

[248] Cf. Bronshtein et al. (2007), p. 694, (14.56).

[249] Cf. Rowland and Weisstein (2009).

For a function $f = g(z)/h(z)$, where h has a simple zero in z_0, the residue can be determined with

$$\mathrm{Res}_{z_0}(f) = \frac{g(z_0)}{h'(z_0)}. \tag{4.171}$$

4.5.6.1.3 Partitions

A partition p of a positive integer m is a way to express m as a sum of positive integers in non-decreasing order. A partition p of m will be denoted by $p \prec m$. A partition p can be indicated by $p = 1^{e_1}, 2^{e_2}, \ldots, m^{e_m}$, where e_i is the frequency of the number i in the partition. The number of summands of p is expresses by $|p|$, which is the sum $|p| = e_1 + e_2 + \ldots + e_m$. The notation $\hat{p}$ indicates the partition which results if each summand of a partition p is increased by 1. This means that for $p \prec m$ the partition $\hat{p}$ refers to a specific partition of $m + |p|$.[250]

Example

- For $m = 5$, there exist seven partitions $p \prec m$: $p \prec m = \{1+1+1+1+1,\ 1+1+1+2,\ 1+2+2,\ 1+1+3,\ 2+3,\ 1+4,\ 5\}$. Thus, a concrete partition for $m = 5$ is $p = 3+1+1$.
- This partition can also be denoted by $p = 1^{e_1} 2^{e_2} \ldots m^{e_m} = 1^2 3^1$, leading to $e_1 = 2,\ e_2 = 0,\ e_3 = 1,\ e_4 = 0$, and $e_5 = 0$. Thus, the number m results from: $m = 1 \cdot e_1 + 2 \cdot e_2 + \ldots + m \cdot e_m = 1 \cdot 2 + 3 \cdot 1 = 5$.
- The number of summands of this partition is $\left|p = 1^2 3^1\right| = e_1 + e_2 + \ldots + e_m = 2 + 1 = 3$.
- The partition $\hat{p}$ appendant to the partition $p = 3+1+1$ is $\hat{p} = 4+2+2$, which is a specific partition of $m + |p| = 5 + 3 = 8$.

4.5.6.2 Determination of the Derivatives

4.5.6.2.1 Derivatives of the Distribution Function

Proposition. *The derivatives of the distribution function of losses* $F_{Y+\lambda Z}(y) = \mathbb{P}(\tilde{Y} + \lambda \tilde{Z} \le y)$ *at* $\lambda = 0$ *are given as*[251]

$$\left.\frac{\partial^m}{\partial \lambda^m} F_{Y+\lambda Z}(y)\right|_{\lambda=0} = (-1)^m \frac{d^{m-1}}{dy^{m-1}} \left(\mathbb{E}\left(\tilde{Z}^m \mid \tilde{Y} = y\right) f_Y(y)\right). \tag{4.172}$$

[250] Cf. Wilde (2003), p. 3 f.

[251] See Martin and Wilde (2002), p. 124 f., and Wilde (2003), p. 2 f.

Proof. Using the definition of the Laplace transform (4.160) and recognizing that the loss $\tilde{L} = \tilde{Y} + \lambda\tilde{Z}$ cannot go below zero so that the probability density function is $f_{Y+\lambda Z}(y) = 0$ for all $y < 0$, we get for the Laplace transform of $f_{Y+\lambda Z}(y)$

$$\mathcal{L}\{f_{Y+\lambda Z}(y)\} = \int\limits_{y=-0}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy = \int\limits_{y=-\infty}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy. \tag{4.173}$$

With the definition of the expectation operator

$$\mathbb{E}\left(g\left(\tilde{X}\right)\right) = \int\limits_{x=-\infty}^{\infty} g(x) f_X(x)dx, \tag{4.174}$$

(4.173) is equivalent to

$$\mathcal{L}\{f_{Y+\lambda Z}(y)\} = \int\limits_{y=-\infty}^{\infty} \mathrm{e}^{-sy} f_{Y+\lambda Z}(y)dy = \mathbb{E}\left(\mathrm{e}^{-s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right). \tag{4.175}$$

Applying the definition of the inverse Laplace transform (4.161) and using the moment generating function M of $\tilde{Y} + \lambda\tilde{Z}$, which is defined as[252]

$$M_{Y+\lambda Z}(s) = \mathbb{E}\left(\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right), \tag{4.176}$$

the probability density function equals[253]

$$\begin{aligned} f_{Y+\lambda Z}(y) &= \mathcal{L}^{-1}\{\mathcal{L}\{f_{Y+\lambda Z}(y)\}\} = \mathcal{L}^{-1}\{M_{Y+\lambda Z}(-s)\} \\ &= \frac{1}{2\pi i} \int\limits_{s=c-i\infty}^{c+i\infty} M_{Y+\lambda Z}(s)\mathrm{e}^{-sy} ds. \end{aligned} \tag{4.177}$$

Thus, the derivatives of the probability density function at $\lambda = 0$ can be determined using the approach

$$\left.\frac{\partial^m}{\partial\lambda^m} f_{Y+\lambda Z}(y)\right|_{\lambda=0} = \left.\frac{1}{2\pi i} \int\limits_{s=c-i\infty}^{c+i\infty} \frac{\partial^m}{\partial\lambda^m} M_{Y+\lambda Z}(s)\mathrm{e}^{-sy} ds\right|_{\lambda=0}. \tag{4.178}$$

[252] Cf. Billingsley (1995), p. 146 ff., for details about moment generating functions.
[253] Cf. Miller and Childers (2004), p. 118.

Applying definition (4.176), we obtain for the derivatives of M

$$\begin{aligned}\left.\frac{\partial^m M_{Y+\lambda Z}(s)}{\partial \lambda^m}\right|_{\lambda=0} &= \left.\frac{\partial^m}{\partial \lambda^m}\mathbb{E}\left(\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0} \\ &= \left.\mathbb{E}\left(\frac{\partial^m}{\partial \lambda^m}\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0} \\ &= \left.\mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\left(\tilde{Y}+\lambda\tilde{Z}\right)}\right)\right|_{\lambda=0} \\ &= \mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\tilde{Y}}\right). \end{aligned} \tag{4.179}$$

With (4.179) and $s^m\mathrm{e}^{s\left(\tilde{Y}-y\right)} = (-1)^m\frac{\partial^m}{\partial y^m}\mathrm{e}^{s\left(\tilde{Y}-y\right)}$, (4.178) is equivalent to

$$\begin{aligned}\left.\frac{\partial^m}{\partial \lambda^m}f_{Y+\lambda Z}(y)\right|_{\lambda=0} &= \frac{1}{2\pi i}\int_{s=c-i\infty}^{c+i\infty}\mathbb{E}\left(s^m\tilde{Z}^m\mathrm{e}^{s\tilde{Y}}\right)e^{-sy}ds \\ &= \mathbb{E}\left(\frac{1}{2\pi i}\tilde{Z}^m\int_{s=c-i\infty}^{c+i\infty}s^m\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds\right) \\ &= (-1)^m\frac{d^m}{dy^m}\mathbb{E}\left(\tilde{Z}^m\frac{1}{2\pi i}\int_{s=c-i\infty}^{c+i\infty}\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds\right). \end{aligned} \tag{4.180}$$

According to (4.164), Dirac's delta function can be written as

$$\delta(t) = \frac{1}{2\pi i}\int_{s=c-i\infty}^{c+i\infty}1\cdot\mathrm{e}^{st}ds, \tag{4.181}$$

which leads to

$$\delta\left(\tilde{Y}-y\right) = \frac{1}{2\pi i}\int_{s=c-i\infty}^{c+i\infty}1\cdot\mathrm{e}^{s\left(\tilde{Y}-y\right)}ds \tag{4.182}$$

for $t=\tilde{Y}-y$. Hence, (4.180) is equivalent to

$$\left.\frac{\partial^m}{\partial \lambda^m}f_{Y+\lambda Z}(y)\right|_{\lambda=0} = (-1)^m\frac{d^m}{dy^m}\mathbb{E}\left(\tilde{Z}^m\delta\left(\tilde{Y}-y\right)\right). \tag{4.183}$$

With $\mathbb{E}[\tilde{Z}^m \delta(\tilde{Y} - y)] = \mathbb{E}[\tilde{Z}^m \mid \tilde{Y} = y] \cdot f_Y(y)$, the derivatives of the distribution function result after integration of (4.183):

$$\frac{\partial^m}{\partial \lambda^m} F_{Y+\lambda Z}(y)\bigg|_{\lambda=0} = (-1)^m \frac{d^{m-1}}{dy^{m-1}} \left(\mathbb{E}\left(\tilde{Z}^m \mid \tilde{Y} = y\right) f_Y(y)\right), \tag{4.184}$$

which is proposition (4.172). In order to determine the derivatives of the quantile $d^m q/d\lambda^m$, the implicit derivatives of $F(q(\lambda),\lambda) - \alpha = 0$ with $F(q(\lambda),\lambda) := F_{\tilde{Y}+\lambda\tilde{Z}}(q_\alpha(\tilde{Y}+\lambda\tilde{Z})) = \mathbb{P}\left(\tilde{Y}+\lambda\tilde{Z} \le q_\alpha(\tilde{Y}+\lambda\tilde{Z})\right)$ will be calculated in the following.

4.5.6.2.2 Implicit Derivatives: Complex Residue Form

Consider a function $G(z,w)$ of two variables $z,w \in \mathbb{C}$. Suppose there exists an analytic function $w = w(z)$ in a region around a pole $z = z_0$, such that $G(z,w(z)) = 0$. The first derivative dw/dz can be determined as follows:[254]

$$\begin{aligned} 0 &= \frac{\partial G}{\partial z} + \frac{\partial G}{\partial w} \cdot \frac{dw}{dz} \\ \Leftrightarrow \frac{dw}{dz} &= -\frac{\partial G/\partial z}{\partial G/\partial w} =: -\frac{G_z}{G_w}. \end{aligned} \tag{4.185}$$

Proposition. *For $G_w(z_0,w_0) \neq 0$, the derivatives $d^m w/dz^m$ are given as*

$$\frac{d^m w}{dz^m} = -\mathrm{Res}_{w_0}\left[\frac{\partial^{m-1}}{\partial z^{m-1}}\left(\frac{G_z(z,w)}{G(z,w)}\right)\bigg|_{z=z_0}\right]. \tag{4.186}$$

Proof. According to (4.186), the first derivative is

$$\frac{dw}{dz} = -\mathrm{Res}_{w_0}\left[\left(\frac{G_z(z,w)}{G(z,w)}\right)\bigg|_{z=z_0}\right] = -\mathrm{Res}_{w_0}\left[\frac{G_z(z_0,w)}{G(z_0,w)}\right]. \tag{4.187}$$

As z_0 is a pole of G and $G(z_0,w) = 0$, an application of (4.171) leads to

$$\frac{dw}{dz} = -\mathrm{Res}_{w_0}\left[\frac{G_z(z_0,w)}{G(z_0,w)}\right] = -\frac{G_z}{G_w}, \tag{4.188}$$

[254] For ease of notation, the derivatives $\partial G/\partial z$ and $\partial G/\partial w$ will be abbreviated to G_z and G_w, respectively. The function G is not associated with a random variable, so confusion should not arise with respect to the similar notation $F_{Y+\lambda Z}(y)$, where the subscript of the distribution function F denotes the corresponding random variable.

which is equal to (4.185). This shows that the formula is correct for $m = 1$.

Applying the residue theorem (4.169)

$$\sum_{\mu=1}^{m} a_{-1}(z_\mu) = \frac{1}{2\pi i} \oint_C f(z)dz \tag{4.189}$$

and recognizing that there is only a singularity at $z = z_0$ leads to

$$\frac{dw}{dz} = -\text{Res}_{w_0}\left[\left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0}\right] = -\frac{1}{2\pi i}\oint_C \left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw. \tag{4.190}$$

Differentiating and applying the residue theorem again results in

$$\begin{aligned}\frac{d^m w}{dz^m} &= \frac{\partial^{m-1}}{\partial z^{m-1}}\left(-\frac{1}{2\pi i}\oint_C \left.\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw\right)\\ &= -\frac{1}{2\pi i}\oint_C \left.\frac{\partial^{m-1}}{\partial z^m}\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0} dw\\ &= -\text{Res}_{w_0}\left[\left.\frac{\partial^{m-1}}{\partial z^m}\frac{G_z(z,w)}{G(z,w)}\right|_{z=z_0}\right],\end{aligned} \tag{4.191}$$

which is the proposition presented in (4.186). This result is a generalization of the Lagrange inversion theorem.[255]

4.5.6.2.3 Implicit Derivatives: Combinatorial Form

In order to express the implicit derivatives (4.191) in combinatorial form, *Faà di Bruno's formula* will be used. According to this formula, the following equation holds for a function $g = g(y)$ with $y = y(x)$:[256]

$$\frac{d^m g}{dx^m} = \sum_{p \prec m} \alpha_p \frac{d^{|p|}g}{dy^{|p|}}\frac{d^p y}{dx^p}, \tag{4.192}$$

[255] Cf. Wilde (2003), p. 7.

[256] See Abramowitz and Stegun (1972), Sect. 24.1.2(C). The notation $p \prec m$ indicates that p is a partition of m, cf. Sect. 4.5.6.1.3.

with $\alpha_p = \frac{m!}{(1!)^{e_1} \cdot e_1! \cdot \ldots \cdot (m!)^{e_m} \cdot e_m!}$, $\frac{d^{|p|}g}{dy^{|p|}}$ as ordinary $|p|$th derivative, and

$$\frac{d^p y}{dx^p} := \left(\frac{dy}{dx}\right)^{e_{p1}} \cdot \left(\frac{d^2 y}{dx^2}\right)^{e_{p2}} \cdot \ldots \cdot \left(\frac{d^m y}{dx^m}\right)^{e_{pm}} = \prod_{i=1}^{m} \left(\frac{d^i y}{dx^i}\right)^{e_{pi}}. \tag{4.193}$$

Proposition. *Equation (4.191) is equivalent to*

$$\frac{d^m w}{dz^m} = \sum_{p \prec m, u \prec s \leq |p|-1} \alpha_p \alpha_{\hat{u}} \frac{(-1)^{|p|+|u|}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} G_w{}^{-|p|-|u|} \frac{\partial^{\hat{u}} G}{\partial w^{\hat{u}}} \frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}} \frac{\partial^p G}{\partial z^p}\bigg|_{z,w=0}. \tag{4.194}$$

Proof. For ease of notation, it will be assumed that $z_0 = w_0 = 0$, so that $G(0,0) = 0$. With $\partial \ln G / \partial z = G_z / G$, (4.191) is equivalent to

$$\frac{d^m w}{dz^m} = -\mathrm{Res}_{w_0}\left[\frac{\partial^{m-1}}{\partial z^{m-1}} \left(\frac{G_z}{G}\right)\bigg|_{z=0}\right] = -\mathrm{Res}_{w_0}\left[\frac{\partial^m}{\partial z^m} \ln G\bigg|_{z=0}\right]. \tag{4.195}$$

The mth derivative of $\ln G$ can be calculated using Faà di Bruno's formula:

$$\begin{aligned}\frac{\partial^m}{\partial z^m} \ln G &= \sum_{p \prec m} \alpha_p \frac{d^{|p|} \ln G}{dG^{|p|}} \frac{\partial^p G}{\partial z^p} = \sum_{p \prec m} \alpha_p \frac{d^{|p|-1}}{dG^{|p|-1}} \left(\frac{1}{G}\right) \frac{\partial^p G}{\partial z^p} \\ &= \sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot (|p|-1)! \cdot G^{-|p|} \cdot G_{z,p},\end{aligned} \tag{4.196}$$

with $\partial^p G / \partial z^p =: G_{z,p}$. This leads to

$$\begin{aligned}\frac{d^m w}{dz^m} &= -\mathrm{Res}_{w_0}\left[\frac{\partial^m}{\partial z^m} \ln G\bigg|_{z=0}\right] \\ &= -\mathrm{Res}_{w_0}\left[\sum_{p \prec m} \alpha_p \cdot (-1)^{|p|-1} \cdot (|p|-1)! \cdot G^{-|p|} \cdot G_{z,p}\bigg|_{z=0}\right].\end{aligned} \tag{4.197}$$

According to (4.170), the residue of a function $h(w)$ in w_0, with w_0 being a pole of order r, can be calculated as

$$\mathrm{Res}_{w_0}[h(w)] = \lim_{w \to w_0} \frac{1}{(r-1)!} \frac{d^{r-1}}{dw^{r-1}} \left((w-w_0)^r \cdot h(w)\right). \tag{4.198}$$

With $r = |p|$, we obtain for the derivative (4.197)

$$\begin{aligned}\frac{d^m w}{dz^m} &= -\mathrm{Res}_{w_0}\left[\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot(|p|-1)!\cdot G^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right]\\ &= -\frac{1}{(|p|-1)!}\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left[w^{|p|}\cdot\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot(|p|-1)!\cdot G^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right]\Bigg|_{w=0}\\ &= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left(\left(\frac{G}{w}\right)^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right)\Bigg|_{w=0}. \end{aligned} \tag{4.199}$$

Using the Leibniz identity for arbitrary-order derivatives of products of functions, we get:[257]

$$\begin{aligned}\frac{d^m w}{dz^m} &= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\frac{\partial^{|p|-1}}{\partial w^{|p|-1}}\left(\left(\frac{G}{w}\right)^{-|p|}\cdot G_{z,p}\Big|_{z=0}\right)\Bigg|_{w=0}\\ &= -\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\sum_{s=0}^{|p|-1}\binom{|p|-1}{s}\\ &\qquad\cdot\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|}\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\big|_{w=0}. \end{aligned} \tag{4.200}$$

As a next step, the derivative $\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|}$ contained in (4.200) will be calculated. Performing a Taylor series expansion of $G(0,w)$ at $w=0$, we have

$$\begin{aligned}G(0,w) &= G(0,0)+\frac{w}{1!}\cdot\frac{\partial}{\partial w}G(0,0)+\frac{w^2}{2!}\cdot\frac{\partial^2}{\partial w^2}G(0,0)+\frac{w^3}{3!}\cdot\frac{\partial^3}{\partial w^3}G(0,0)+\ldots\\ &= 0+w\cdot G_w(0,0)+\sum_{r\geq 2}\frac{w^r}{r!}\cdot\frac{\partial^r}{\partial w^r}G(0,0)\\ &= w\cdot G_w(0,0)+\sum_{r\geq 1}\frac{w^{r+1}}{(r+1)!}\cdot\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)\\ &= w\cdot G_w(0,0)+w\cdot G_w(0,0)\cdot\sum_{r\geq 1}\frac{w^r}{(r+1)!}\cdot\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)\cdot\frac{1}{G_w(0,0)}\\ &= w\cdot G_w(0,0)\cdot\left(1+\sum_{r\geq 1}\frac{w^r}{r!}\cdot\frac{1}{r+1}\cdot\frac{\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)}{\frac{\partial}{\partial w}G(0,0)}\right). \end{aligned} \tag{4.201}$$

[257] See Weisstein (2009b).

Thus, for $G(0,w)/w$, we obtain

$$\begin{aligned}\frac{G(0,w)}{w} &= G_w(0,0)\cdot\left(1+\sum_{r\geq 1}\frac{w^r}{r!}\cdot\frac{1}{r+1}\cdot\frac{\frac{\partial^{r+1}}{\partial w^{r+1}}G(0,0)}{\frac{\partial}{\partial w}G(0,0)}\right)\\ &= G_w(0,0)\cdot\left(1+\sum_{r\geq 1}\frac{w^r}{r!}\cdot\varphi_r\right),\end{aligned}\tag{4.202}$$

with $\varphi_r=\frac{1}{r+1}\cdot\frac{\partial^{r+1}/\partial w^{r+1}G(0,0)}{\partial/\partial wG(0,0)}$. Another application of Faà di Bruno's formula results in:[258]

$$\begin{aligned}\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|} &= G_w^{-|p|}(0,0)\cdot\frac{\partial^s}{\partial w^s}\left(1+\sum_{r\geq 1}\varphi_r\cdot\frac{w^r}{r!}\right)^{-|p|}\\ &= G_w^{-|p|}(0,0)\cdot\sum_{u\prec s}\alpha_u\cdot\varphi_u\cdot(-1)^{|u|}\cdot\frac{(|p|+|u|-1)!}{(|p|-1)!},\end{aligned}\tag{4.203}$$

with[259]

$$\alpha_u\cdot\varphi_u=\frac{s!}{(s+|u|)!}\cdot\alpha_{\hat{u}}\cdot\frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}}G(0,0)\cdot G_w{}^{-|u|}(0,0).\tag{4.204}$$

Applying (4.203) and (4.204) to (4.200) leads to

$$\begin{aligned}\frac{d^m w}{dz^m}&=-\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\sum_{s=0}^{|p|-1}\binom{|p|-1}{s}\frac{\partial^s}{\partial w^s}\left(\frac{G(0,w)}{w}\right)^{-|p|}\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}\\ &=-\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\sum_{s=0}^{|p|-1}\binom{|p|-1}{s}\cdot G_w^{-|p|}(0,0)\cdot\sum_{u\prec s}\frac{s!}{(s+|u|)!}\cdot\alpha_{\hat{u}}\cdot\frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}}G(0,0)\\ &\quad\cdot G_w{}^{-|u|}(0,0)\cdot(-1)^{|u|}\cdot\frac{(|p|+|u|-1)!}{(|p|-1)!}\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}\\ &=-\sum_{p\prec m}\alpha_p\cdot(-1)^{|p|-1}\cdot\sum_{s=0}^{|p|-1}\binom{|p|-1}{s}\cdot\sum_{u\prec s}\alpha_{\hat{u}}\cdot(-1)^{|u|}\cdot G_w^{-|p|-|u|}(0,0)\\ &\quad\cdot\frac{s!\cdot(|p|+|u|-1)!}{(s+|u|)!\cdot(|p|-1)!}\cdot\frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}}G(0,0)\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\Big|_{w=0}.\end{aligned}\tag{4.205}$$

[258]Cf. Wilde (2003), p. 8.

[259]The relation between a partition u and $\hat{u}$ is explained in Sect. 4.5.6.1.3.

Summarizing the sums, using $(-1)\cdot(-1)^{|p|-1}\cdot(-1)^{|u|}=(-1)^{|p|+|u|}$, and

$$
\begin{aligned}
&\binom{|p|-1}{s}\cdot\frac{s!}{(|p|-1)!}\cdot\frac{(|p|+|u|-1)!}{(s+|u|)!}\\
&=\frac{(|p|-1)!}{s!\cdot(|p|-1-s)!}\cdot\frac{s!}{(|p|-1)!}\cdot\frac{(|p|+|u|-1)!}{(s+|u|)!}\\
&=\frac{(|p|+|u|-1)!}{(|p|-1-s)!\cdot(s+|u|)!},
\end{aligned}
\tag{4.206}
$$

(4.205) can be simplified to

$$
\begin{aligned}
\frac{d^m w}{dz^m}=&\sum_{p\prec m,u\prec s\le|p|-1}\alpha_p\cdot\alpha_{\hat{u}}\cdot(-1)^{|p|+|u|}\cdot\frac{(|p|+|u|-1)!}{(|p|-1-s)!\cdot(s+|u|)!}\\
&\cdot G_w^{-|p|-|u|}(0,0)\cdot\frac{\partial^{\hat{u}}}{\partial w^{\hat{u}}}G(0,0)\\
&\cdot\frac{\partial^{|p|-1-s}}{\partial w^{|p|-1-s}}\left(G_{z,p}(0,w)\right)\bigg|_{w=0},
\end{aligned}
\tag{4.207}
$$

which concludes the proof.

4.5.6.2.4 Completion of the Derivation

Application of (4.207) can be used to determine the derivatives of a quantile, which will be calculated subsequently. With $F(q(\lambda),\lambda)-\alpha=0=G(w(z),z)$, the derivatives are given as

$$
\frac{d^m q}{d\lambda^m}\bigg|_{\lambda=0}=\frac{d^m w}{dz^m}\bigg|_{z=0},
\tag{4.208}
$$

where the right-hand side can be determined with (4.207). The derivatives of G contained in (4.207) can be calculated with (4.172):

$$
\begin{aligned}
\frac{\partial^{r+s}G}{\partial w^r\partial z^s}\bigg|_{z=0}&=\frac{\partial^{r+s}F}{\partial y^r\partial\lambda^s}\bigg|_{\lambda=0}=\frac{\partial^r}{\partial y^r}\left(\frac{\partial^s F}{\partial\lambda^s}\right)\bigg|_{\lambda=0}\\
&=\frac{d^r}{dy^r}\left((-1)^s\frac{d^{s-1}}{dy^{s-1}}\left(\mathbb{E}\big(\tilde{Z}^s\mid\tilde{Y}=y\big)f_Y(y)\right)\right)\\
&=(-1)^s\frac{d^{r+s-1}}{dy^{r+s-1}}\left(\mathbb{E}\big(\tilde{Z}^s\mid\tilde{Y}=y\big)f_Y(y)\right)\\
&=(-1)^s\frac{d^{r+s-1}}{dy^{r+s-1}}\left(\mu_{s,c}f\right),
\end{aligned}
\tag{4.209}
$$

where we define $\mu_{s,c} := \mathbb{E}(\tilde{Z}^s \mid \tilde{Y} = y)$ and $f := f_Y(y)$ for convenience. Using definition (4.193) for the pth derivative with $p \prec m$, this leads to

$$\left.\frac{\partial^p G}{\partial z^p}\right|_{z=0} = \left.\prod_{i=1}^{m}\left(\frac{\partial^i G}{\partial z^i}\right)^{e_{pi}}\right|_{z=0} = \prod_{i=1}^{m}\left((-1)^i \frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right)^{e_{pi}} = (-1)^m \prod_{i=1}^{m}\left(\frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right)^{e_{pi}}. \tag{4.210}$$

Similarly the $\hat{u}$th derivative can be determined with $u \prec s$. It has to be considered that for each partition u the elements of the corresponding partition $\hat{u}$ are increased by 1. Thus, the smallest number is 2 and the largest is $s+1$. Hence, we obtain

$$\begin{aligned}\frac{\partial^{\hat{u}} G}{\partial w^{\hat{u}}} &= \prod_{i=2}^{s+1}\left(\frac{\partial^i G}{\partial w^i}\right)^{e_{\hat{u}i}} = \prod_{i=2}^{s+1}\left(\frac{\partial^i G}{\partial w^i}\right)^{e_{u(i-1)}} = \prod_{i=1}^{s}\left(\frac{\partial^{i+1} G}{\partial w^{i+1}}\right)^{e_{ui}} \\ &= \prod_{i=1}^{s}\left(\frac{\partial^{i+1} F}{\partial y^{i+1}}\right)^{e_{ui}} = \prod_{i=1}^{s}\left(\frac{d^i f}{dy^i}\right)^{e_{ui}}.\end{aligned} \tag{4.211}$$

Furthermore, we have $G_w = dF/dy = f$ and $(-1)^{|p|+|u|} \cdot f^{|p|+|u|} = (-f)^{|p|+|u|}$. Using these formulas, we finally get for (4.207) or (4.208):

$$\begin{aligned}\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} = (-1)^m \Bigg[\sum_{p\prec m, u\prec s\le |p|-1} \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} \cdot (-f)^{-|p|-|u|} \cdot \left(\prod_{i=1}^{s}\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right) \\ \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^{m}\left[\frac{d^{i-1}(\mu_{i,c}f)}{dy^{i-1}}\right]^{e_{pi}}\right)\Bigg]_{y=q_\alpha(\tilde{Y})},\end{aligned} \tag{4.212}$$

which is the formula for arbitrary derivatives of VaR. Written without abbreviations this is

$$\begin{aligned}\left.\frac{d^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m \Bigg[\sum_{p\prec m, u\prec s\le |p|-1} \frac{\alpha_p \alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!} \cdot (-f_Y(y))^{-|p|-|u|} \\ &\cdot \left(\prod_{i=1}^{s}\left[\frac{d^i f_Y(y)}{dy^i}\right]^{e_{ui}}\right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \\ &\cdot \left(\prod_{i=1}^{m}\left[\frac{d^{i-1}\left(\mathbb{E}(\tilde{Z}^m \mid \tilde{Y}=y) f_Y(y)\right)}{dy^{i-1}}\right]^{e_{pi}}\right)\Bigg]_{y=q_\alpha(\tilde{Y})},\end{aligned} \tag{4.213}$$

with $\alpha_p = \frac{m!}{(1!)^{e_{p1}} e_{p,1}! \cdot \ldots \cdot (m!)^{e_{p,m}} e_{p,m}!}$.

4.5.7 Determination of the First Five Derivatives of VaR

The general form of the mth derivative of VaR is given by (4.213). Subsequently, the first five derivatives will be determined with this formula. For each derivative, we have summands for all partitions $p \prec m$ and $u \prec s \leq |p| - 1$. For the considered cases $1 \leq m \leq 5$, the following partitions $p \prec m$ exist:

$$\begin{aligned}
&p \prec 1 = \{1^1\};\\
&p \prec 2 = \{1^2,\ 2^1\};\\
&p \prec 3 = \{1^3,\ 1^1 2^1,\ 3^1\};\\
&p \prec 4 = \{1^4,\ 1^2 2^1,\ 2^2,\ 1^1 3^1,\ 4^1\};\\
&p \prec 5 = \{1^5,\ 1^3 2^1,\ 1^1 2^2,\ 1^2 3^1,\ 2^1 3^1,\ 1^1 4^1,\ 5^1\}.
\end{aligned} \tag{4.214}$$

By construction, the expectation of the unsystematic loss is zero:

$$\mu_{1,c}(y) = \mathbb{E}\left(\tilde{Z}^1 \mid \tilde{Y} = y\right) = 0, \tag{4.215}$$

which is called the "granularity adjustment condition". Consequently, for all partitions with $e_{p1} \neq 0$, the summands of (4.213) are zero, too:

$$\begin{aligned}
\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}} &= 0^{e_{p1}} \cdot \prod_{i=2}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}}\\
&= \begin{cases} \prod_{i=2}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}}\right]^{e_{pi}} & \text{if } e_{p1} = 0,\\ 0 & \text{if } e_{p1} \neq 0. \end{cases}
\end{aligned} \tag{4.216}$$

Hence, the only relevant partitions $p \prec m$ of (4.214) with non-zero terms and the corresponding numbers $|p|$ are given as[260]

$$\begin{aligned}
&p \prec 1 = \{1^1\} && \text{with } \left|p = 1^1\right| = 1,\\
&p \prec 2 = \{2^1\} && \text{with } \left|p = 2^1\right| = 1,\\
&p \prec 3 = \{3^1\} && \text{with } \left|p = 3^1\right| = 1,\\
&p \prec 4 = \{4^1,\ 2^2\} && \text{with } \left|p = 4^1\right| = 1,\ \left|p = 2^2\right| = 2,\\
&p \prec 5 = \{5^1,\ 2^1 3^1\} && \text{with } \left|p = 5^1\right| = 1,\ \left|p = 2^1 3^1\right| = 2.
\end{aligned} \tag{4.217}$$

For the associated terms

---

[260] In order to demonstrate that the resulting formula is also valid for $m = 1$, the summand for partition $\{1^1\}$, which equals zero due to argument (4.216), is still considered.

$$\alpha_p = \frac{m!}{(1!)^{e_{p1}} e_{p,1}! \cdot \ldots \cdot (m!)^{e_{p,m}} e_{p,m}!}, \tag{4.218}$$

we obtain

$$\begin{aligned}
\alpha_{1^1} &= \frac{1!}{(1!)^1 \cdot 1!} = 1, \\
\alpha_{2^1} &= \frac{2!}{(2!)^1 \cdot 1!} = 1, \\
\alpha_{3^1} &= \frac{3!}{(3!)^1 \cdot 1!} = 1, \\
\alpha_{4^1} &= \frac{4!}{(4!)^1 \cdot 1!} = 1, \quad \alpha_{2^2} = \frac{4!}{(2!)^2 \cdot 2!} = \frac{24}{8} = 3, \\
\alpha_{5^1} &= \frac{5!}{(5!)^1 \cdot 1!} = 1, \quad \alpha_{2^1 3^1} = \frac{5!}{(2!)^1 \cdot 1! \cdot (3!)^1 \cdot 1!} = \frac{120}{12} = 10.
\end{aligned} \tag{4.219}$$

According to (4.217), we only have $|p| = 1$ and $|p| = 2$, leading to the following partitions $u \prec s \leq |p| - 1$:

$$\begin{aligned}
|p| = 1: &\quad u \prec (s = 0) = \{0\}, \\
|p| = 2: &\quad u \prec \{s = 0,\ s = 1\} = \left\{0,\ 1^1\right\}.
\end{aligned} \tag{4.220}$$

As we have one summand for each $p \prec m$ and $u \prec s \leq (|p| - 1)$, we obtain one summand for $m = 1,\ 2,\ 3$ and three summands for $m = 4,\ 5$:

$$\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} = \begin{cases} (I), & \text{if } m = 1,\ 2,\ 3, \\ (I) + (II) + (III), & \text{if } m = 4,\ 5, \end{cases} \tag{4.221}$$

where the summands are determined with the following variables:

$$\begin{aligned}
&(I) \quad m = 1,\ \ldots,\ 5: p = m^1, && |p| = 1,\ u \prec (s = 0) = \{0\}, \\
&(II) \quad \left.\begin{matrix} m = 4: & p = 2^2, \\ m = 5: & p = 2^1 3^1, \end{matrix}\right\} && |p| = 2,\ u \prec (s = 0) = \{0\}, \\
&(III) \quad \left.\begin{matrix} m = 4: & p = 2^2, \\ m = 5: & p = 2^1 3^1, \end{matrix}\right\} && |p| = 2,\ u \prec (s = 1) = \left\{1^1\right\}.
\end{aligned} \tag{4.222}$$

The first summand (I), with $p = m^1,\ |p| = 1,\ s = 0,\ u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{pm} = 1$, and $e_{pi} = 0$ for all $i \neq m$, equals:[261]

[261]For ease of notation, the arguments $\lambda = 0$ of the left-hand as well as $y = q_\alpha(\tilde{Y})$ at the right-hand side are omitted.

$$
\begin{aligned}
(I) &= \frac{\alpha_p \alpha_{\hat{u}} (|p| + |u| - 1)!}{(s + |u|)!(|p| - 1 - s)!} (-f)^{-|p|-|u|} \\
&\quad \cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= \frac{1 \cdot 1 \cdot (1 + 0 - 1)!}{(0 + 0)!(1 - 1 - 0)!} (-f)^{-1-0} \left(\prod_{i=1}^{0} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{1-1-0}}{dy^{1-1-0}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= -\frac{1}{f} \cdot \frac{d^{m-1}\left(\mu_{m,c} f\right)}{dy^{m-1}}.
\end{aligned} \tag{4.223}
$$

For $m = 4$, the second summand (II.[4]), with values $p = 2^2,\ |p| = 2,\ s = 0,$ $u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{p2} = 2$, and $e_{pi} = 0$ for all $i \neq 2$, is equivalent to

$$
\begin{aligned}
II.[4] &= \frac{\alpha_p \alpha_{\hat{u}} (|p| + |u| - 1)!}{(s + |u|)!(|p| - 1 - s)!} (-f)^{-|p|-|u|} \\
&\quad \cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= \frac{3 \cdot 1 \cdot (2 + 0 - 1)!}{(0 + 0)!(2 - 1 - 0)!} (-f)^{-2-0} \left(\prod_{i=1}^{0} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{2-1-0}}{dy^{2-1-0}} \left(\prod_{i=1}^{4} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= 3 \cdot \frac{1}{f^2} \cdot \frac{d}{dy} \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right]^2.
\end{aligned} \tag{4.224}
$$

For $m = 5$, we have $p = 2^1 3^1,\ |p| = 2,\ s = 0,\ u = 0, |u| = 0,\ \hat{u} = 1^1$, $e_{p2} = 1,\ e_{p3} = 1,$ and $e_{pi} = 0$ for all $i \neq 2, 3$, leading to

$$
\begin{aligned}
II.[5] &= \frac{\alpha_p \alpha_{\hat{u}} (|p| + |u| - 1)!}{(s + |u|)!(|p| - 1 - s)!} (-f)^{-|p|-|u|} \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \\
&\quad \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= \frac{10 \cdot 1 \cdot (2 + 0 - 1)!}{(0 + 0)!(2 - 1 - 0)!} (-f)^{-2-0} \left(\prod_{i=1}^{0} \left[\frac{d^i f}{dy^i} \right]^{e_{uj}} \right) \cdot \frac{d^{2-1-0}}{dy^{2-1-0}} \left(\prod_{i=1}^{5} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
&= 10 \cdot \frac{1}{f^2} \cdot \frac{d}{dy} \left(\left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right] \left[\frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right] \right).
\end{aligned} \tag{4.225}
$$

The third summand for $m=4$ ($III.[4]$), with $p=2^2,\ |p|=2,\ s=1,\ u=1^1,$ $|u|=1,\ \hat{u}=2^1,\ e_{p2}=2,\ e_{pi}=0$ for all $i\neq 2$, and $e_{u1}=1$ equals

$$
\begin{aligned}
III.[4] =& \frac{\alpha_p \alpha_{\hat{u}} (|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}(-f)^{-|p|-|u|} \\
&\cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& \frac{3\cdot 1\cdot (2+1-1)!}{(1+1)!(2-1-1)!}(-f)^{-2-1} \left(\prod_{i=1}^{1} \left[\frac{d^i f}{dy^i} \right]^{1} \right) \cdot \frac{d^{2-1-1}}{dy^{2-1-1}} \left(\prod_{i=1}^{4} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& -3 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right]^2 .
\end{aligned} \tag{4.226}
$$

For $m=5$, we have $p=2^1 3^1,\ |p|=2,\ s=1,\ u=1^1,\ |u|=1,\ \hat{u}=2^1,\ e_{p2}=1,$ $e_{p3}=1,\ e_{pi}=0$ for all $i\neq 2,3$, and $e_{u1}=1$. Hence, we get

$$
\begin{aligned}
III.[5] =& \frac{\alpha_p \alpha_{\hat{u}} (|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}(-f)^{-|p|-|u|} \\
&\cdot \left(\prod_{i=1}^{s} \left[\frac{d^i f}{dy^i} \right]^{e_{ui}} \right) \cdot \frac{d^{|p|-1-s}}{dy^{|p|-1-s}} \left(\prod_{i=1}^{m} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& \frac{10\cdot 1\cdot (2+1-1)!}{(1+1)!(2-1-1)!}(-f)^{-2-1} \left(\prod_{i=1}^{1} \left[\frac{d^i f}{dy^i} \right]^{1} \right) \cdot \frac{d^{2-1-1}}{dy^{2-1-1}} \left(\prod_{i=1}^{5} \left[\frac{d^{i-1}\left(\mu_{i,c} f\right)}{dy^{i-1}} \right]^{e_{pi}} \right) \\
=& -10 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d\left(\mu_{2,c} f\right)}{dy} \right] \cdot \left[\frac{d^2\left(\mu_{3,c} f\right)}{dy^2} \right].
\end{aligned} \tag{4.227}
$$

Summing up the relevant elements from (4.223) to (4.227) and multiplying by $(-1)^m$ leads to

$$
\left.\frac{dq}{d\lambda}\right|_{\lambda=0} = (-1)^1 \cdot \left(-\frac{1}{f}\right) \cdot \frac{d^{1-1}\left(\mu_{1,c} f\right)}{dy^{1-1}} = \mu_{1,c} = 0, \tag{4.228}
$$

$$
\left.\frac{d^2 q}{d\lambda^2}\right|_{\lambda=0} = (-1)^2 \cdot \left(-\frac{1}{f}\right) \cdot \frac{d^{2-1}\left(\mu_{2,c} f\right)}{dy^{2-1}} = -\frac{1}{f} \cdot \frac{d\left(\mu_{2,c} f\right)}{dy}, \tag{4.229}
$$

$$
\left.\frac{d^3 q}{d\lambda^3}\right|_{\lambda=0} = (-1)^3 \cdot \left(-\frac{1}{f}\right) \cdot \frac{d^{3-1}\left(\mu_{3,c} f\right)}{dy^{3-1}} = \frac{1}{f} \cdot \frac{d^2\left(\mu_{3,c} f\right)}{dy^2}, \tag{4.230}
$$

$$\left.\frac{d^4 q}{d\lambda^4}\right|_{\lambda=0} = (-1)^4 \cdot \left[\left(-\frac{1}{f}\right) \cdot \frac{d^{4-1}(\mu_{4,c}f)}{dy^{4-1}} + 3 \cdot \frac{1}{f^2} \cdot \frac{d}{dy}\left(\frac{d(\mu_{2,c}f)}{dy}\right)^2 \right.$$
$$\left. -3 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left(\frac{d(\mu_{2,c}f)}{dy}\right)^2\right]$$
$$= \left(-\frac{1}{f}\right) \cdot \left(\frac{d^3(\mu_{4,c}f)}{dy^3} - 3 \cdot \frac{d}{dy}\left[\frac{1}{f}\left(\frac{d(\mu_{2,c}f)}{dy}\right)^2\right]\right), \tag{4.231}$$

and

$$\left.\frac{d^5 q}{d\lambda^5}\right|_{\lambda=0} = (-1)^5 \cdot \left[\left(-\frac{1}{f}\right) \cdot \frac{d^{5-1}(\mu_{5,c}f)}{dy^{5-1}} + 10 \cdot \frac{1}{f^2} \cdot \frac{d}{dy}\left(\left[\frac{d(\mu_{2,c}f)}{dy}\right]\left[\frac{d^2(\mu_{3,c}f)}{dy^2}\right]\right)\right.$$
$$\left. -10 \cdot \frac{1}{f^3} \cdot \frac{df}{dy} \cdot \left[\frac{d(\mu_{2,c}f)}{dy}\right] \cdot \left[\frac{d^2(\mu_{3,c}f)}{dy^2}\right]\right]$$
$$= \frac{1}{f} \cdot \left[\frac{d^4(\mu_{5,c}f)}{dy^4} - 10 \cdot \frac{d}{dy}\left(\frac{1}{f} \cdot \frac{d(\mu_{2,c}f)}{dy}\,\frac{d^2(\mu_{3,c}f)}{dy^2}\right)\right]. \tag{4.232}$$

Comparing these terms, we find that the derivatives for $m = 1, \ldots, 5$ can be written as

$$\left.\frac{d^m q}{d\lambda^m}\right|_{\lambda=0} = (-1)^m\left(-\frac{1}{f}\right)\left[\frac{d^{m-1}(\mu_{m,c}f)}{dy^{m-1}} - \kappa(m)\right.$$
$$\left. \cdot \frac{d}{dy}\left(\frac{1}{f} \cdot \frac{d(\mu_{2,c}f)}{dy}\,\frac{d^{m-3}(\mu_{m-2,c}f)}{dy^{m-3}}\right)\right] \tag{4.233}$$

or without abbreviations as

$$\left.\frac{d^m VaR_\alpha(\tilde{Y} + \lambda\tilde{Z})}{d\lambda^m}\right|_{\lambda=0} = (-1)^m\left(-\frac{1}{f_Y(y)}\right)\left[\frac{d^{m-1}\left(\mu_m(\tilde{Z}\,|\,\tilde{Y} = y)f_Y(y)\right)}{dy^{m-1}}\right.$$
$$- \kappa(m) \cdot \frac{d}{dy}\left(\frac{1}{f_Y(y)} \cdot \frac{d\left(\mu_2(\tilde{Z}\,|\,\tilde{Y} = y)f_Y(y)\right)}{dy}\right.$$
$$\left.\left. \cdot \frac{d^{m-3}\left(\mu_{m-2}(\tilde{Z}\,|\,\tilde{Y} = y)f_Y(y)\right)}{dy^{m-3}}\right)\right]_{y=q_\alpha(\tilde{Y})}, \tag{4.234}$$

with $\kappa(1) = \kappa(2) = 0,\ \kappa(3) = 1,\ \kappa(4) = 3$, and $\kappa(5) = 10$, which is the result of Wilde (2003).

4.5.8 Order of the Derivatives of VaR

For any $m \in \mathbb{N}$, the $(m+1)$th element of the Taylor series can be written as[262]

$$\frac{\lambda^m}{m!}\left[\frac{\partial^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{\partial\lambda^m}\right]_{\lambda=0} = g \circ \left(\frac{\lambda^m}{m!}\sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{Z}\,|\,\tilde{Y}=y\right]\right)^{e_{pi}}\right)\Bigg|_{y=q_\alpha(\tilde{Y})}, \quad (4.235)$$

with g being a function that is independent of the number of credits n. With μ_i as the ith moment about the origin and η_i as the ith moment about the mean, it is possible to write[263]

$$\begin{aligned}
\lambda^m\sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{Z}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})} &= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\lambda\tilde{Z}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})} \\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\tilde{L}-\mathbb{E}\left(\tilde{L}|\tilde{x}\right)|\tilde{x}=x\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})} \\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\mu_i\left[\left(\tilde{L}|\tilde{x}=x\right)-\mathbb{E}\left(\tilde{L}|\tilde{x}=x\right)\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})} \\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}|\tilde{x}=x\right]\right)^{e_{pi}}\Big|_{x=q_{1-\alpha}(\tilde{x})} \\
&= \sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}|\tilde{Y}=y\right]\right)^{e_{pi}}\Big|_{y=q_\alpha(\tilde{Y})} \qquad (4.236)
\end{aligned}$$

for each m. Thus, the derivatives are given as

$$\frac{\lambda^m}{m!}\left[\frac{\partial^m VaR_\alpha(\tilde{Y}+\lambda\tilde{Z})}{\partial\lambda^m}\right]_{\lambda=0} = g \circ \left(\frac{1}{m!}\sum_{p\prec m}\prod_{i=1}^{m}\left(\eta_i\left[\tilde{L}\,|\,\tilde{Y}=y\right]\right)^{e_{pi}}\right)\Bigg|_{y=q_\alpha(\tilde{Y})}. \quad (4.237)$$

---

[262] Cf. (4.213). The notation $g \circ y$ means that a function g is composed with y.

[263] To illustrate that the first identity holds, an example will be demonstrated for m = 5:

$$\begin{aligned}
\lambda\cdot\sum_{p\prec 5}\prod_{i=1}^{5}\left(\mu_i(\tilde{Z})\right)^{e_{pi}} &= \lambda\cdot\Big(\mu_5(\tilde{Z})+\mu_4(\tilde{Z})\cdot\mu_1(\tilde{Z})+\mu_3(\tilde{Z})\cdot\left(\mu_1(\tilde{Z})\right)^2 \\
&\quad +\mu_3(\tilde{Z})\cdot\mu_2(\tilde{Z})+\mu_2(\tilde{Z})\cdot\left(\mu_1(\tilde{Z})\right)^3+\mu_2(\tilde{Z})^2\cdot\mu_1(\tilde{Z})+\left(\mu_1(\tilde{Z})\right)^5\Big) \\
&= \mu_5(\lambda\tilde{Z})+\mu_4(\lambda\tilde{Z})\cdot\mu_1(\lambda\tilde{Z})+\mu_3(\lambda\tilde{Z})\cdot\left(\mu_1(\lambda\tilde{Z})\right)^2 \\
&\quad +\mu_3(\lambda\tilde{Z})\cdot\mu_2(\lambda\tilde{Z})+\mu_2(\lambda\tilde{Z})\cdot\left(\mu_1(\lambda\tilde{Z})\right)^3+\mu_2(\lambda\tilde{Z})^2\cdot\mu_1(\lambda\tilde{Z}) \\
&\quad +\left(\mu_1(\lambda\tilde{Z})\right)^5.
\end{aligned}$$

Furthermore, see (4.9) for the switch between the systematic loss y and the systematic factor x.

Due to[264]

$$\eta_i\left(\tilde{L} \,|\, \tilde{x} = x\right) = \eta_i^{\,*}(x) \cdot \sum_{j=1}^{n} w_j^{\,i} \leq \eta_i^{\,*}(x) \cdot \left(\frac{b}{a}\right)^i \cdot \frac{1}{n^{i-1}} = O\left(\frac{1}{n^{i-1}}\right),$$

with $0 < a \leq EAD_i \leq b$ for all i, and revisiting (4.235) and (4.236), it is straightforward to see that only for $m = 3$ and $m = 4$ there exist terms which are at maximum of order $O(1/n^2)$:

$$\begin{aligned}
&\sum_{p \prec 3} \prod_{i=1}^{3} \left(\eta_i\left[\tilde{L} \,|\, \tilde{Y} = y\right]\right)^{e_{pi}} = \eta_3\left[\tilde{L} \,|\, \tilde{Y} = y\right] = O\left(\frac{1}{n^2}\right),\\
&\sum_{p \prec 4} \prod_{i=1}^{4} \left(\eta_i\left[\tilde{L} \,|\, \tilde{Y} = y\right]\right)^{e_{pi}} = \eta_4\left[\tilde{L} \,|\, \tilde{Y} = y\right] + \left(\eta_2\left[\tilde{L} \,|\, \tilde{Y} = y\right]\right)^2 = O\left(\frac{1}{n^3}\right) + O\left(\frac{1}{n^2}\right).
\end{aligned} \tag{4.238}$$

All terms with higher derivatives of VaR are at least of Order $O(1/n^3)$.

4.5.9 VaR-Based Second-Order Granularity Adjustment for a Normally Distributed Systematic Factor

For convenience, the summands of the second-order granularity add-on Δl_2 will be calculated separately:

$$\begin{aligned}
\Delta l_2 &= \frac{1}{6\varphi} \frac{d}{dx} \left(\frac{1}{d\mu_{1,c}/dx} \frac{d}{dx} \left[\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx} \right] \right) \\
&\quad + \frac{1}{8\varphi} \frac{d}{dx} \left[\frac{1}{\varphi} \frac{1}{d\mu_{1,c}/dx} \left(\frac{d}{dx} \left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx} \right] \right)^2 \right] \Bigg|_{x=\Phi^{-1}(1-\alpha)} \\
&=: \Delta l_{2,1} + \Delta l_{2,2} \Big|_{x=\Phi^{-1}(1-\alpha)}.
\end{aligned} \tag{4.239}$$

[264] See (4.14).

The term $\Delta l_{2,1}$ equals

$$\begin{aligned}\Delta l_{2,1} &= \frac{1}{6}\left[\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\frac{1}{\varphi}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)+\frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\frac{d^2}{dx^2}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\right]\\ &= \frac{1}{6}\left[\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\left(\underbrace{\frac{1}{\varphi}\frac{d}{dx}\left(\eta_{3,c}\varphi\right)}_{=:A}\frac{1}{d\mu_{1,c}/dx}+\eta_{3,c}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right)\right.\\ &\quad\left.+\frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\frac{d}{dx}\left[\underbrace{\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}}_{=:B}+\underbrace{\eta_{3,c}\varphi\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)}_{=:C}\right]\right].\end{aligned} \tag{4.240}$$

For the calculation, we need the first and second derivative of the density function φ. As the systematic factor is assumed to be normally distributed, we have

$$\varphi = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \tag{4.241}$$

$$\frac{d\varphi}{dx} = (-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = -x\,\varphi, \tag{4.242}$$

$$\frac{d^2\varphi}{dx^2} = (-1)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x(-x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = (x^2-1)\varphi. \tag{4.243}$$

Furthermore, we need the derivative

$$\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right) = -\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}. \tag{4.244}$$

Herewith, the term A form (4.240) can easily be calculated:

$$A = \frac{1}{\varphi}\frac{d}{dx}\left(\eta_{3,c}\varphi\right) = \frac{d\eta_{3,c}}{dx}+\frac{\eta_{3,c}}{\varphi}\frac{d\varphi}{dx} = \frac{d\eta_{3,c}}{dx}-\eta_{3,c}x. \tag{4.245}$$

Furthermore, dB/dx is equal to

$$
\begin{aligned}
\frac{dB}{dx} &= \frac{d}{dx}\left(\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}\right)\\
&= \frac{d^2}{dx^2}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx} + \frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\\
&= \frac{d}{dx}\left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{1}{d\mu_{1,c}/dx} + \left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\left(-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\
&= \left(\frac{d^2\eta_{3,c}}{dx^2}\varphi + 2\frac{d\eta_{3,c}}{dx}\frac{d\varphi}{dx} + \eta_{3,c}\frac{d^2\varphi}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx}\\
&\quad - \left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}.
\end{aligned}
\tag{4.246}
$$

Similarly, dC/dx is equivalent to

$$
\begin{aligned}
\frac{dC}{dx} &= \frac{d}{dx}\left(\eta_{3,c}\varphi\left(-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right)\\
&= -\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2} - \eta_{3,c}\varphi\frac{d}{dx}\left(\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\
&= \left(-\frac{d\eta_{3,c}}{dx}\varphi - \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\\
&\quad - \eta_{3,c}\varphi\left(\frac{\left(d\mu_{1,c}/dx\right)^2\left(d^3\mu_{1,c}/dx^3\right) - 2\left(d\mu_{1,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4}\right).
\end{aligned}
\tag{4.247}
$$

Using these terms, $\Delta l_{2,1}$ results in

$$
\begin{aligned}
\Delta l_{2,1} = \frac{1}{6}\Bigg[&-\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\left(\frac{d\eta_{3,c}/dx}{d\mu_{1,c}/dx} - \frac{\eta_{3,c}x}{d\mu_{1,c}/dx} - \eta_{3,c}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\
&+ \frac{1}{d\mu_{1,c}/dx}\frac{1}{\varphi}\Bigg[\left(\frac{d^2\eta_{3,c}}{dx^2}\varphi + 2\frac{d\eta_{3,c}}{dx}\frac{d\varphi}{dx} + \eta_{3,c}\frac{d^2\varphi}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx}\\
&- 2\left(\frac{d\eta_{3,c}}{dx}\varphi + \eta_{3,c}\frac{d\varphi}{dx}\right)\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\\
&-\eta_{3,c}\varphi\left(\frac{\left(d\mu_{1,c}/dx\right)^2\left(d^3\mu_{1,c}/dx^3\right) - 2\left(d\mu_{1,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4}\right)\Bigg]\Bigg].
\end{aligned}
\tag{4.248}
$$

Applying the derivatives of φ from (4.242) and (4.243) leads to

$$\begin{aligned}\Delta l_{2,1} =& \frac{1}{6}\Bigg[-3\frac{\left(d\eta_{3,c}/dx\right)\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^3}+3\frac{\eta_{3,c}x\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^3}+3\eta_{3,c}\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^4}\\
&+\frac{d^2\eta_{3,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}-2x\frac{d\eta_{3,c}/dx}{\left(d\mu_{1,c}/dx\right)^2}+\frac{\eta_{3,c}(x^2-1)}{\left(d\mu_{1,c}/dx\right)^2}-\eta_{3,c}\frac{d^3\mu_{1,c}/dx^3}{\left(d\mu_{1,c}/dx\right)^3}\Bigg]\\
=&\frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\Bigg[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\\
&+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\Bigg]. \qquad (4.249)\end{aligned}$$

Henceforward, the summand $\Delta l_{2,2}$ will be simplified:

$$\begin{aligned}\Delta l_{2,2} &= \frac{1}{8\varphi}\frac{d}{dx}\left[\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left(\frac{d}{dx}\left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right]\right)^2\right]\\
&=\frac{1}{8\varphi}\frac{d}{dx}\left(\frac{\varphi}{d\mu_{1,c}/dx}\underbrace{\left(\frac{1}{\varphi}\frac{d}{dx}\left[\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right]\right)^2}_{*}\right). \qquad (4.250)\end{aligned}$$

The term (*) is the negative twice of the first-order granularity adjustment, so that we can use the resulting equation (4.18). This leads to

$$\begin{aligned}\Delta l_{2,2} &= \frac{1}{8\varphi}\frac{d}{dx}\left(\frac{\varphi}{d\mu_{1,c}/dx}\left[-\frac{x\,\eta_{2,c}}{d\mu_{1,c}/dx}+\frac{d\eta_{2,c}/dx}{d\mu_{1,c}/dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]^2\right)\\
&=\frac{1}{8}\Bigg[\underbrace{\frac{1}{\varphi}\frac{d}{dx}\left(\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\right)}_{=:(I)}\left(-x\,\eta_{2,c}+\frac{d\eta_{2,c}}{dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)^2\\
&\quad+\frac{1}{\left(d\mu_{1,c}/dx\right)^3}\underbrace{\frac{d}{dx}\left(\left[-x\,\eta_{2,c}+\frac{d\eta_{2,c}}{dx}-\frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]^2\right)}_{=:(II)}\Bigg]. \qquad (4.251)\end{aligned}$$

Using the derivative of a normal distribution $d\varphi/dx = -x\,\varphi$, the term ($I$) is equivalent to

$$
\begin{aligned}
(I) &= \frac{1}{\varphi}\frac{d}{dx}\left(\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\right) \\
&= \frac{1}{\varphi}\frac{d\varphi}{dx}\frac{1}{\left(d\mu_{1,c}/dx\right)^3} + \frac{d}{dx}\left(\frac{1}{\left(d\mu_{1,c}/dx\right)^3}\right) \\
&= \frac{-x}{\left(d\mu_{1,c}/dx\right)^3} - 3\frac{\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^4}.
\end{aligned} \tag{4.252}
$$

Term (II) can be written as

$$
\begin{aligned}
(II) &= \frac{d}{dx}\left(\left[-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]^2\right) \\
&= 2\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\Bigg(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2} \\
&\quad - \frac{d}{dx}\left(\eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\right)\frac{1}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\Bigg) \\
&= 2\left(-x\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\Bigg(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2} \\
&\quad - \frac{d\eta_{2,c}}{dx}\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx} + \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\Bigg).
\end{aligned} \tag{4.253}
$$

Using these expressions, $\Delta l_{2,2}$ from (4.251) is equal to

$$
\begin{aligned}
\Delta l_{2,2} = \frac{1}{8}\Bigg[&\left(\frac{-x}{\left(d\mu_{1,c}/dx\right)^3} - 3\frac{\left(d^2\mu_{1,c}/dx^2\right)}{\left(d\mu_{1,c}/dx\right)^4}\right)\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)^2 \\
&+ \frac{2}{\left(d\mu_{1,c}/dx\right)^3}\left(-x\,\eta_{2,c} + \frac{d\eta_{2,c}}{dx} - \frac{\eta_{2,c}d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\Bigg(-\eta_{2,c} - x\frac{d\eta_{2,c}}{dx} + \frac{d^2\eta_{2,c}}{dx^2} \\
&- \frac{d\eta_{2,c}}{dx}\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx} - \eta_{2,c}\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx} + \eta_{2,c}\frac{d^2\mu_{1,c}}{dx^2}\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\Bigg)\Bigg],
\end{aligned} \tag{4.254}
$$

which leads to

$$\begin{aligned}\Delta l_{2,2} = &\frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.\\ &+2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.\\ &\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]. \end{aligned} \tag{4.255}$$

Adding the terms $\Delta l_{2,1}$ and $\Delta l_{2,2}$ together results in

$$\begin{aligned}\Delta l_2 = &\frac{1}{6\left(d\mu_{1,c}/dx\right)^2}\left[\eta_{3,c}\left(x^2-1-\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}+\frac{3x\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}+\frac{3\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right)\right.\\ &\left.+\frac{d\eta_{3,c}}{dx}\left(-2x-\frac{3\left(d^2\mu_{1,c}/dx^2\right)}{d\mu_{1,c}/dx}\right)+\frac{d^2\eta_{3,c}}{dx^2}\right]\\ &+\frac{1}{8\left(d\mu_{1,c}/dx\right)^3}\left[\left(-x-3\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\left(\eta_{2,c}\left[-x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]+\frac{d\eta_{2,c}}{dx}\right)^2\right.\\ &+2\left(\eta_{2,c}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d\eta_{2,c}}{dx}\right)\left(\eta_{2,c}\left[1+\frac{d^3\mu_{1,c}/dx^3}{d\mu_{1,c}/dx}-\frac{\left(d^2\mu_{1,c}/dx^2\right)^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\right.\\ &\left.\left.\left.+\frac{d\eta_{2,c}}{dx}\left[x+\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right]-\frac{d^2\eta_{2,c}}{dx^2}\right)\right]\right|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.256}$$

4.5.10 Third Conditional Moment of Losses

Subsequently, the third conditional moment of the portfolios loss about the mean, $\eta_{3,c} = \eta_3(\tilde{L}\,|\,\tilde{x}=x)$, shall be expressed in terms of the moments of separated factors $\widetilde{LGD}_i$ and $1_{\{\tilde{D}_i\}}$. With

$$\begin{aligned}\eta_{3,c} &= \eta_3\left(\tilde{L}\,|\,\tilde{x}=x\right)\\ &= \eta_3\left(\sum_{i=1}^{n} w_i\cdot\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right)\\ &= \sum_{i=1}^{n} w_i^3\cdot\eta_3\left(\widetilde{LGD}_i\cdot 1_{\{\tilde{D}_i\}}\,|\,\tilde{x}=x\right), \end{aligned} \tag{4.257}$$

which is due to the conditional independence property, we need to determine $\eta_3(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x})$. In general, the third moment about the mean is equal to

$$\begin{aligned}
\eta_3(\tilde{X}) &= \mathbb{E}\left(\left[\tilde{X} - \mathbb{E}(\tilde{X})\right]^3\right) \\
&= \mathbb{E}\left[\tilde{X}^3 - 3\tilde{X}^2\mathbb{E}(\tilde{X}) + 3\tilde{X}\mathbb{E}^2(\tilde{X}) - \mathbb{E}^3(\tilde{X})\right] \\
&= \mathbb{E}\left(\tilde{X}^3\right) - 3\mathbb{E}\left(\tilde{X}^2\right)\mathbb{E}(\tilde{X}) + 3\mathbb{E}(\tilde{X})\mathbb{E}^2(\tilde{X}) - \mathbb{E}^3(\tilde{X}) \\
&= \mathbb{E}\left(\tilde{X}^3\right) - 3\mathbb{E}\left(\tilde{X}^2\right)\mathbb{E}(\tilde{X}) + 2\mathbb{E}^3(\tilde{X}).
\end{aligned} \tag{4.258}$$

Thus, the conditional moment $\eta_3(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x})$ can be written as

$$\begin{aligned}
\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right) = {}& \mathbb{E}\left(\left[\widetilde{LGD} \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right]^3\right) \\
& - 3\,\mathbb{E}\left(\left[\widetilde{LGD} \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right]^2\right) \cdot \mathbb{E}\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right) \\
& + 2\,\mathbb{E}^3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right).
\end{aligned} \tag{4.259}$$

Using the conditional independence property again, considering that the LGDs are assumed to be stochastically independent of each other, and with $\mathbb{E}[(1_{\{\tilde{D}_i\}} \,|\, \tilde{x})^i] = \mathbb{E}[(1_{\{\tilde{D}_i\}} \,|\, \tilde{x})] = p(\tilde{x})$, we have

$$\begin{aligned}
\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right) &= \mathbb{E}\left(\left[\widetilde{LGD} \,|\, \tilde{x}\right]^3\right)p(\tilde{x}) - 3\,\mathbb{E}\left(\left[\widetilde{LGD} \,|\, \tilde{x}\right]^2\right)\mathbb{E}\left(\widetilde{LGD} \,|\, \tilde{x}\right)p^2(\tilde{x}) \\
&\quad + 2\,\mathbb{E}^3\left(\widetilde{LGD} \,|\, \tilde{x}\right)p^3(\tilde{x}) \\
&= \mathbb{E}\left(\widetilde{LGD}^3\right)p(\tilde{x}) - 3\,\mathbb{E}\left(\widetilde{LGD}^2\right)\mathbb{E}\left(\widetilde{LGD}\right)p^2(\tilde{x}) \\
&\quad + 2\,\mathbb{E}^3\left(\widetilde{LGD}\right)p^3(\tilde{x}).
\end{aligned} \tag{4.260}$$

With the abbreviations $ELGD = \mathbb{E}(\widetilde{LGD})$, $VLGD = \mathbb{V}(\widetilde{LGD})$ as well as $SLGD = \eta_3(\widetilde{LGD})$ and using (4.258) again, we obtain

$$\mathbb{E}\left(\widetilde{LGD}^2\right) = ELGD^2 + VLGD, \tag{4.261}$$

$$\begin{aligned}
\mathbb{E}\left(\widetilde{LGD}^3\right) &= SLGD + 3(ELGD^2 + VLGD)ELGD - 2\,ELGD^3 \\
&= ELGD^3 + 3\,ELGD \cdot VLGD + SLGD.
\end{aligned} \tag{4.262}$$

Consequently, (4.260) is equivalent to

$$\eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x}\right) = \left(ELGD^3 + 3\,ELGD \cdot VLGD + SLGD\right)p(\tilde{x}) \\ - 3\left(ELGD^3 + ELGD \cdot VLGD\right)p^2(\tilde{x}) + 2\,ELGD^3 p^3(\tilde{x}). \tag{4.263}$$

Thus, the conditional moment of the portfolio loss (4.257) can finally be written as

$$\begin{aligned}\eta_{3,c} &= \sum_{i=1}^{n} w_i{}^3 \cdot \eta_3\left(\widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}} \,|\, \tilde{x} = x\right) \\ &= \sum_{i=1}^{n} w_i{}^3 \left[\left(ELGD_i{}^3 + 3 \cdot ELGD_i \cdot VLGD_i + SLGD_i\right) \cdot p_i(x)\right. \\ &\quad \left. -3 \cdot \left(ELGD_i{}^3 + ELGD_i \cdot VLGD_i\right) \cdot p_i{}^2(x) + 2 \cdot ELGD_i{}^3 \cdot p_i{}^3(x)\right].\end{aligned} \tag{4.264}$$

4.5.11 Difference Between the VaR Definitions

For the case of homogeneous credits and with $LGD = 1$, the possible realizations of losses are

$$l \in \left\{0,\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n},\ 1\right\}, \tag{4.265}$$

which implies

$$\mathbb{P}\left[\tilde{L} \le l\right] = \mathbb{P}\left[\tilde{L} < (l + 1/n)\right]. \tag{4.266}$$

If we define $l_2 := l_1 + 1/n$, we get

$$\begin{aligned}VaR_\alpha^{(-)}\left(\tilde{L}\right) &= \sup\left\{l_1 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} \le l_1\right] < \alpha\right\} \\ &= \sup\left\{l_1 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < \left(l_1 + \frac{1}{n}\right)\right] < \alpha\right\} \\ &= \sup\left\{\left(l_2 - \frac{1}{n}\right) \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l_2\right] < \alpha\right\} \\ &= \sup\left\{l_2 \in \mathbb{R} \,|\, \mathbb{P}\left[\tilde{L} < l_2\right] < \alpha\right\} - \frac{1}{n} \\ &= VaR_\alpha^{(+)}\left(\tilde{L}\right) - \frac{1}{n}.\end{aligned} \tag{4.267}$$

4.5.12 Identity of ES Within the Basel Framework

Using the result of the ASRF framework (2.93), the definition of the ES (2.19), the integral representation of the conditional expectation, and the identity of the condition as in (4.9), the ES of the portfolio loss equals

$$\begin{aligned} ES_\alpha^{(\text{Basel})}\big(\tilde{L}\big) &= ES_\alpha\big[\mathbb{E}\big(\tilde{L}\,|\,\tilde{x}\big)\big] \\ &= ES_\alpha\big[\mu_{1,c}(\tilde{x})\big] \\ &= \frac{1}{1-\alpha}\big[\mathbb{E}\big(\mu_{1,c}(\tilde{x})\,|\,\mu_{1,c}(\tilde{x}) \geq q_\alpha\big(\mu_{1,c}(\tilde{x})\big)\big)\big] \\ &= \frac{1}{1-\alpha}\big[\mathbb{E}\big(\mu_{1,c}(\tilde{x})\,|\,\tilde{x} \leq \Phi^{-1}(1-\alpha)\big)\big] \\ &= \frac{1}{1-\alpha}\int_{-\infty}^{\Phi^{-1}(1-\alpha)} \mu_{1,c}(x)\varphi(x)dx. \end{aligned} \tag{4.268}$$

With the conditional independence property as in (2.92), the conditional PD of the Vasicek model (2.66), the integral representation (2.126), and the symmetry of the normal distribution, the ES can be written as

$$\begin{aligned} ES_\alpha^{(\text{Basel})}\big(\tilde{L}\big) &= \frac{1}{1-\alpha}\int_{-\infty}^{\Phi^{-1}(1-\alpha)} \sum_{i=1}^{n} \mathbb{E}\Big(w_i \cdot \widetilde{LGD}_i \cdot 1_{\{D_i\}}\,|\,x\Big)\varphi(x)dx \\ &= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \int_{-\infty}^{\Phi^{-1}(1-\alpha)} p_i(x)\varphi(x)dx \\ &= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi\left(\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_i}\cdot x}{\sqrt{1-\rho_i}}\right)\varphi(x)dx \\ &= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\big(\Phi^{-1}(1-\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\big) \\ &= \frac{1}{1-\alpha}\sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\big(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), \sqrt{\rho_i}\big). \end{aligned} \tag{4.269}$$

4.5.13 Arbitrary Derivatives of ES

According to (2.20), the ES can be written as

$$ES_\alpha\left(\tilde{L}\right) = \frac{1}{1-\alpha}\int_\alpha^1 q^u\left(\tilde{L}\right)du. \tag{4.270}$$

Thus, for continuous distributions, all derivatives of ES can be expressed as

$$\frac{d^m ES_\alpha}{d\lambda^m} = \frac{d^m}{d\lambda^m}\left(\frac{1}{1-\alpha}\int_\alpha^1 q_u du\right) = \frac{1}{1-\alpha}\int_\alpha^1 \frac{d^m q_u}{d\lambda^m}du. \tag{4.271}$$

The derivative of VaR is a function of $f_Y(y)$ and $\mu_{i,c}(y)$ evaluated at $q_u(\tilde{Y})$. The substitution $u = F_Y(y)$, so that $du/dy = f_Y(y)$, $y(u=\alpha) = F_Y^{-1}(\alpha) = q_\alpha(\tilde{Y})$, and $y(u=1) = F_Y^{-1}(1) = \infty$, leads to:[265]

$$\left.\frac{d^m ES_\alpha}{d\lambda^m}\right|_{\lambda=0} = \frac{1}{1-\alpha}\int_{u=\alpha}^1 \left.\frac{d^m q_u}{d\lambda^m}\right|_{\lambda=0} du = \frac{1}{1-\alpha}\int_{y=q_\alpha\left(\tilde{Y}\right)}^\infty \left.\frac{d^m q_u}{d\lambda^m}\right|_{\lambda=0} f_Y dy, \tag{4.272}$$

where the expression resulting from the derivative of VaR simply has to be evaluated at y since $q_u(\tilde{Y}) = y$. Using the derivatives of VaR from (4.212), this leads to

$$\left.\frac{d^m ES_\alpha}{d\lambda^m}\right|_{\lambda=0} = \frac{1}{1-\alpha}\int_{y=q_\alpha\left(\tilde{Y}\right)}^\infty (-1)^m \left[\sum_{p\prec m, u\prec s\le|p|-1} \frac{\alpha_p\alpha_{\hat{u}}(|p|+|u|-1)!}{(s+|u|)!(|p|-1-s)!}\right.$$
$$\left.\cdot(-f)^{-|p|-|u|}\cdot\left(\prod_{i=1}^s\left[\frac{d^i f}{dy^i}\right]^{e_{ui}}\right)\cdot\frac{d^{|p|-1-s}}{dy^{|p|-1-s}}\left(\prod_{i=1}^m\left[\frac{d^{i-1}\left(\mu_{i,c}f\right)}{dy^{i-1}}\right]^{e_{pi}}\right)\right] f\,dy, \tag{4.273}$$

with $\alpha_p = \dfrac{m!}{(1!)^{e_{p1}}e_{p,1}!\cdot\ldots\cdot(m!)^{e_{p,m}}e_{p,m}!}$.

[265] Cf. Wilde (2003), p. 11.

4.5.14 Determination of the First Five Derivatives of ES

Instead of solving the integral (4.272) for each of the derivatives of VaR (4.228)–(4.232), we will directly evaluate the integral for the first five derivatives. Using the expression for the first five derivatives of VaR (4.233), we obtain

$$\begin{aligned}\left.\frac{d^m ES}{d\lambda^m}\right|_{\lambda=0} &= \frac{1}{1-\alpha}\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty} \frac{d^m q}{d\lambda^m} f_Y dy \\ &= \frac{1}{1-\alpha}\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty} (-1)^m\left(-\frac{1}{f}\right)\left[\frac{d^{m-1}\left(\mu_{m,c}f\right)}{dy^{m-1}}\right. \\ &\quad \left. -\kappa(m)\cdot\frac{d}{dy}\left(\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\,\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right)\right] f\,dy. \end{aligned} \tag{4.274}$$

This term is equal to

$$\begin{aligned}\left.\frac{d^m ES}{d\lambda^m}\right|_{\lambda=0} &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}\left(-\frac{d^{m-1}\left(\mu_{m,c}f\right)}{dy^{m-1}}\right)dy\right. \\ &\quad \left. +\kappa(m)\cdot\int\limits_{y=q_\alpha(\tilde{Y})}^{\infty}\frac{d}{dy}\left(\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right)dy\right) \\ &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\left[-\frac{d^{m-2}\left(\mu_{m,c}f\right)}{dy^{m-2}}\right]_{q_\alpha(\tilde{Y})}^{\infty}\right. \\ &\quad \left. +\kappa(m)\cdot\left[\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right]_{y=q_\alpha(\tilde{Y})}^{\infty}\right) \\ &= (-1)^m\cdot\frac{1}{1-\alpha}\cdot\left(\frac{d^{m-2}\left(\mu_{m,c}f\right)}{dy^{m-2}}\right. \\ &\quad \left.\left. -\kappa(m)\cdot\left[\frac{1}{f}\cdot\frac{d\left(\mu_{2,c}f\right)}{dy}\cdot\frac{d^{m-3}\left(\mu_{m-2,c}f\right)}{dy^{m-3}}\right]\right)\right|_{y=q_\alpha(\tilde{Y})}, \end{aligned} \tag{4.275}$$

or written without abbreviations as

$$\left.\frac{d^m ES_\alpha\left(\tilde{Y}+\lambda\tilde{Z}\right)}{d\lambda^m}\right|_{\lambda=0} = (-1)^m \cdot \frac{1}{1-\alpha} \cdot \left(\frac{d^{m-2}\left(\mu_m\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-2}}\right.$$
$$\left.\left. - \kappa(m) \cdot \left[\frac{1}{f_Y(y)} \cdot \frac{d\left(\mu_2\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy} \cdot \frac{d^{m-3}\left(\mu_{m-2}\left(\tilde{Z}\,|\,\tilde{Y}=y\right)f_Y(y)\right)}{dy^{m-3}}\right]\right)\right|_{y=q_\alpha\left(\tilde{Y}\right)}, \tag{4.276}$$

with $\kappa(1)=\kappa(2)=0,\ \kappa(3)=1,\ \kappa(4)=3$, and $\kappa(5)=10$. This is the result of Wilde (2003), except that the algebraic signs of Wilde (2003) seem to be wrong.

4.5.15 ES-Based Second-Order Granularity Adjustment for a Normally Distributed Systematic Factor

The summands of the second-order granularity add-on Δl_2 can be expressed as

$$\begin{aligned}\Delta l_2 &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\\ &\quad \left.+\frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2\right|_{x=\Phi^{-1}(1-\alpha)}\\ &=: \Delta l_{2,1}+\Delta l_{2,2}\big|_{x=\Phi^{-1}(1-\alpha)}.\end{aligned} \tag{4.277}$$

Using the derivative of the normal distribution (4.242), the summand $\Delta l_{2,1}$ equals

$$\begin{aligned}\Delta l_{2,1} &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\frac{d}{dx}\left(\frac{\eta_{3,c}\varphi}{d\mu_{1,c}/dx}\right)\\ &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\eta_{3,c}\varphi\right)\frac{1}{d\mu_{1,c}/dx}+\eta_{3,c}\varphi\frac{d}{dx}\left(\frac{1}{d\mu_{1,c}/dx}\right)\right]\\ &= \frac{1}{6(1-\alpha)}\frac{1}{d\mu_{1,c}/dx}\left[\left(\frac{d\eta_{3,c}}{dx}\varphi+\eta_{3,c}\frac{d\varphi}{dx}\right)\frac{1}{d\mu_{1,c}/dx}-\eta_{3,c}\varphi\frac{d^2\mu_{1,c}/dx^2}{\left(d\mu_{1,c}/dx\right)^2}\right]\\ &= \frac{1}{6(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2}\left[\frac{d\eta_{3,c}}{dx}-\eta_{3,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right].\end{aligned} \tag{4.278}$$

Using the same transformations, the summand $\Delta l_{2,2}$ is equivalent to

$$\begin{aligned}\Delta l_{2,2} &= \frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{d}{dx}\left(\frac{\eta_{2,c}\varphi}{d\mu_{1,c}/dx}\right)\right]^2\\ &= \frac{1}{8(1-\alpha)}\frac{1}{\varphi}\frac{1}{d\mu_{1,c}/dx}\left[\frac{1}{d\mu_{1,c}/dx}\left[\frac{d\eta_{2,c}}{dx}\varphi-\eta_{2,c}\varphi\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]\right]^2\\ &= \frac{1}{8(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\left[\frac{d\eta_{2,c}}{dx}-\eta_{2,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2, \end{aligned} \tag{4.279}$$

leading to a second-order adjustment of

$$\begin{aligned}\Delta l_2 = &\frac{1}{6(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^2}\left[\frac{d\eta_{3,c}}{dx}-\eta_{3,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]\\ &+\frac{1}{8(1-\alpha)}\frac{\varphi}{\left(d\mu_{1,c}/dx\right)^3}\left[\frac{d\eta_{2,c}}{dx}-\eta_{2,c}\left(x-\frac{d^2\mu_{1,c}/dx^2}{d\mu_{1,c}/dx}\right)\right]^2\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{4.280}$$

4.5.16 Probability Density Function of the Logit-Normal Distribution

The derivation of the density function is based on the inverse function theorem[266]

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\cdot\left|\frac{dg^{-1}(y)}{dy}\right|. \tag{4.281}$$

For the logit function $\tilde{Y} = e^{\tilde{X}}\big/(1+e^{\tilde{X}})$, we have

$$\begin{aligned} g(x) = y = \frac{e^x}{1+e^x} &= \frac{1}{e^{-x}+1}\\ \Leftrightarrow e^{-x} &= \frac{1}{y}-1\\ \Leftrightarrow g^{-1}(y) = x &= -\ln\left(\frac{1}{y}-1\right) \end{aligned} \tag{4.282}$$

[266]Cf. Appendix 4.5.3.

and

$$\frac{dg^{-1}(y)}{dy} = \frac{d}{dy}\left(-\ln\left(\frac{1}{y}-1\right)\right) = -\frac{1}{\frac{1}{y}-1}\cdot\left(-\frac{1}{y^2}\right) = \frac{1}{y(1-y)}. \tag{4.283}$$

Using the density of a normal distribution (4.82) for f_X and recognizing that y is bounded in the interval [0, 1], we get

$$\begin{aligned} f_Y(y) &= f_X\left(-\ln\left(\frac{1}{y}-1\right)\right)\cdot\left|\frac{1}{y(1-y)}\right| \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(-\ln(1/y-1)-\mu_X)^2}{2\sigma_X^2}\right)\cdot\frac{1}{y(1-y)} \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(\ln(1/y-1)+\mu_X)^2}{2\sigma_X^2}\right)\cdot\frac{1}{y(1-y)}. \end{aligned} \tag{4.284}$$

Chapter 5
Model-Based Measurement of Sector Concentration Risk in Credit Portfolios*

5.1 Fundamentals and Research Questions on Sector Concentration Risk

As demonstrated in Chap. 2, within the ASRF model it is assumed that there exists only one single risk factor that influences the defaults of all loans in the portfolio (assumption B). Thus, industry-specific or geographical effects are neglected, which can lead to an inappropriate capital requirement for real-world portfolios if this is measured on the basis of a single-factor model like the IRB Approach of Pillar 1. Against this background, banks are demanded to measure concentration risks and "explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under Pillar 2"[267] of Basel II, but it is not specified how this should be done. Although there exist some models that explicitly deal with the measurement of sector concentration risk, these are mostly not consistent with Pillar 1 of Basel II – sometimes within the derivation and sometimes within the implementation. Consequently, it remains unclear if or how much additional regulatory capital is needed regarding risk concentrations. However, this issue is not only relevant from a regulatory perspective. Generally, it is not worthwhile to have a major gap between the regulatory and the "true" economic capital. A homogenization of these values is one goal of the new Capital Accord and would simplify the management of the credit portfolio.

In order to measure sector concentration risk consistent with the Basel II framework, it has to be reconsidered that the IRB Approach was calibrated on well-diversified bank portfolios.[268] Thus, the additional capital requirement concerning

*The main results of this section comply with Gürtler et al. (2010).

[267] BCBS (2005a), § 773.

[268] Cf. Sect. 3.3.

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sector concentrations has to take this specific calibration of the model used for calculating the Pillar 1 capital requirement into consideration. Consequently, only banks with a lower diversification across sectors than these well-diversified banks should assess additional capital under Pillar 2. As data on the characteristics of these well-diversified portfolios is not publicly available, it is not obvious how we can use them as a reference portfolio in order to modify and adjust the existing models on sector concentration risk to achieve consistency to the Basel framework. Furthermore, comparative analyses on models which are able to measure sector concentration risk are scarce. Against this background, we address the following questions:

- How can the existing approaches be modified and adjusted to be consistent with the Basel framework? Is the risk measure Value at Risk problematic when dealing with sector concentration risk?
- Which methods are capable of measuring concentration risk and how good do they perform in comparison? What are the advantages and disadvantages of these methods?

Subsequently, we propose a methodology how multi-factor models can be used in a way that is consistent with the Basel II framework. This can be seen as expanding the validity of the Basel formula from the inner region of Fig. 3.2 to the whole region. To our best knowledge, this is the first work that deals with this problem.[269] Furthermore, we analyze the models of Pykhtin (2004), Cespedes et al. (2006), and Düllmann (2006), which are designed to measure sector concentration risk. We implement our multi-factor setting for these models and use the risk measure ES instead of the VaR, which leads to some new approximation formulas. Based on this, we compare the accuracy and runtime of the different models within a simulation study. Except the rather brief analysis of Düllmann (2007), this is the first comparison of different approaches concerning sector concentration risk. In this context, we also use our framework to test whether the lack of coherency of the widespread used VaR is relevant in connection with the measurement of concentration risk.[270] Since the non-coherency of the VaR is typically only illustrated in contrived portfolio examples, we analyze the relevance of this issue in more realistic settings within our simulation study.

[269]The multi-factor model of Cespedes et al. (2006) is also specified against the background of the regulatory capital formula. However, within the deriviation of their formulas, the authors assume the regulatory capital requirement to be the upper barrier of risk, which is not consistent with the view of supervisors that we presented in Sect. 3.3 and especially in Fig. 3.2. Cf. Sect. 5.2.3 for details regarding this issue.

[270]Cf. Sect. 2.2.3.

5.2 Incorporation of Sector Concentrations Using Multi-Factor Models

5.2.1 *Structure of Multi-Factor Models and Basel II-Consistent Parameterization Through a Correlation Matching Procedure*

To obtain a more realistic modeling of correlated defaults in a credit portfolio, we will introduce a typical *multi-factor model*. In such a model, the dependence structure between obligors is not driven by one global systematic risk factor but by sector specific risk factors. Additionally, the group of obligors is divided into S sectors. Hereby, a suitable sector assignment is important,[271] i.e. asset correlations shall be high within a sector and low between different sectors. In contrast to the single-factor model, in which the correlation structure of each firm is completely described by ρ, in a multi-factor model we distinguish between an *inter-sector correlation* ρ_{Inter} and an *intra-sector correlation* ρ_{Intra}. The inter-sector correlation describes the correlation between the sector factors and the intra-sector correlation characterizes the sensitivity of the asset return to the corresponding sector factor. Thus, the asset return of obligor i in sector s can be represented by[272]

$$\tilde{a}_{s,i} = \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\text{Intra},i}} \cdot \tilde{\xi}_i, \tag{5.1}$$

where $\tilde{x}_s$ is the sector risk factor (with $s = 1, \ldots, S$), and $\tilde{\xi}_i$ stands for the idiosyncratic factor. The variables $\tilde{x}_s$ and $\tilde{\xi}_i$ are normally distributed variables with mean zero and standard deviation one that are independent among each other. Since the sector risk factors $\tilde{x}_s$ are potentially dependent random variables that are difficult to deal with,[273] we make use of the possibility to present the sector risk factors as a combination of independently and standard normally distributed factors $\tilde{z}_k$ ($k = 1, \ldots, K$)

$$\tilde{x}_s = \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k \quad \text{with} \sum_{k=1}^{K} \alpha_{s,k}^2 = 1, \tag{5.2}$$

---

[271] As shown by Morinaga and Shiina (2005), an assignment of borrowers to the wrong sectors usually leads to a higher estimation error than a non-optimal sector definition.

[272] In order to allow for negative intra-sector correlations, the factor loading could also be written as r_i instead of $\sqrt{\rho_{\text{Intra},i}}$. However, it is economically reasonable to assume that there is a positive relationship between the asset return of an obligor and the corresponding industry-sector. Thus, the chosen notation should be no practical limitation.

[273] Concretely, the independence of the risk factors is essential for the derivation of the Pykhtin-model in Sect. 5.2.2.

in which the factor weights $\alpha_{s,k}$ are calculated via a Cholesky decomposition of the inter-sector correlation matrix.[274] Hence, the inter-sector correlation is given as

$$\rho_{s,t}^{\text{Inter}} := \text{Corr}(\tilde{x}_s, \tilde{x}_t) = \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k}. \tag{5.3}$$

From (5.1) and (5.2), the asset correlation between obligors i in sector s and obligor j in sector t is given by

$$\text{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right) = \begin{cases} 1 & \text{if } s = t \text{ and } i = j, \\ \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} & \text{if } s = t \text{ and } i \neq j, \\ \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum\limits_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} & \text{if } s \neq t. \end{cases} \tag{5.4}$$

Obligors in the same sector are highly correlated with one another when their intra-sector correlation is high. The correlation of obligors in different sectors also depends on the factor weights, which are derived from the inter-sector correlation. Hence, the dependence structure in the multi-factor model is completely described by the intra- and inter-sector correlations.

Taking (2.8) into account, the portfolio loss distribution can be written as

$$\tilde{L} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD_{s,i} \cdot 1_{\{\tilde{a}_{s,i} < \Phi^{-1}(PD_{s,i})\}}, \tag{5.5}$$

where n_s is the number of obligors in sector s. The portfolio loss distribution can be determined numerically with Monte Carlo simulations. The procedure is in principle the same as described in Sect. 2.4 in context of the Vasicek one-factor model. In each simulation run, the sector factors as well as the idiosyncratic factor of each obligor are randomly generated. Herewith, the asset return is calculated according to (5.1). If $\tilde{a}_{s,i}$ is less than a threshold given by $\Phi^{-1}(PD_i)$, obligor i defaults. The portfolio loss is determined with (5.5) by summing up the exposure weights w_i multiplied by LGD_i of each defaulted credit. To get a good approximation of the "true" loss distribution, we choose 500,000 runs for our subsequent Monte Carlo simulations. After running the simulation and sorting the loss outcomes, we get the portfolio loss distribution. The ES at a given confidence level α can be calculated with (2.47).

To calibrate the multi-factor model, most variables can be chosen identically to the single-factor model. The only difference is the correlation structure that

[274] This approach is a common mathematical method to generate correlated random variables and leads to the identical number of independent risk factors $\tilde{z}_k$ and dependent sector factors $\tilde{x}_s$, that is K equals S. Another common method to determine independent risk factors is the principal component analysis, which leads to a reduced number of risk factors.

generally consists of inter- and intra-sector correlations as described above. The matrix of inter-sector correlations is usually derived from historical default rates or from equity correlations between industry sectors. The intra-sector correlations can be derived from historical default rates, too. The problem of a derivation based on historical default rates is that there are not always enough observations to get stable results. This is even more problematic if it is assumed (like in Basel II) that the correlation and the PD are interdependent. Furthermore, the results from the multi-factor model would normally not be consistent with Basel II because the correlation structure is completely different. Thus, it would not be possible to identify (consistent with Pillar 1 of Basel II) whether there is need for additional regulatory capital under Pillar 2.

For both reasons, the intra-sector correlations could be chosen analogously to the Basel II formula

$$\rho_{\text{Basel}} = 0.12 \cdot \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}} + 0.24 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}}\right) \tag{5.6}$$

for corporates. This is what Cespedes et al. (2006) did in their analyses. But this assumption is critical for the following reason: The validity of this formula for the intra-sector correlations is equivalent to the statement that the regulatory capital calculated via the formula of Pillar 1 is an upper barrier of the true risk. This property in turn is only fulfilled if either only one sector exists or if all sectors are perfectly correlated. In all other cases there is an effect of sector diversification, which leads to a lower capital requirement compared to the Basel framework. Beyond, the Basel Committee does not intend the Basel II correlation formula to exclusively reflect the intra-sector correlation. Instead, the framework is calibrated on well-diversified portfolios, as demonstrated in Fig. 3.2, implying that the correlation formula is chosen in a way that the single-factor model leads to a good approximation of the "true" risk based on the full correlation structure in a multi-factor model. Cespedes et al. (2006) have already recognized this criticism and have mentioned that it should be possible to use some scaling up for the intra-sector correlations and the resulting capital. However, their calculations are based on the formula above.

Alternatively, the intra-sector correlation could be chosen in a way that the regulatory capital RC can be matched with the economic capital EC^{mf}, which is simulated for a well-diversified portfolio within a multi-factor model. Therefore, we define the "*implicit intra-sector correlation*" $\rho_{\text{Intra}}^{(\text{Implied})}$ by

$$EC^{\text{mf}}\left(\rho_{\text{Inter}}, \rho_{\text{Intra}}^{(\text{Implied})}\right) = RC(\rho_{\text{Basel}}). \tag{5.7}$$

Unfortunately, the portfolios for which the calibration was done by the Basel Committee including the assumed inter-sector correlation structure are not publicly available. Thus, at first we have to choose a concrete inter-sector correlation and determine the implicit intra-sector correlation for some hypothetical, well-diversified

Table 5.1 Inter-sector correlation structure based on MSCI industry indices (in %)[a]

Sector	A	B	C1	C2	C3	D	E	F	H	I	J
A: Energy	100	50	42	34	45	46	57	34	10	31	69
B: Materials		100	87	61	75	84	62	30	56	73	66
C1: Capital goods			100	67	83	92	65	32	69	82	66
C2: Comm. svs. and supplies				100	58	68	40	8	50	60	37
C3: Transportation					100	83	68	27	58	77	67
D: Consumer discretionary						100	76	21	69	81	66
E: Consumer staples							100	33	46	56	66
F: Health care								100	15	24	46
H: Information technology									100	75	42
I: Telecommunication services										100	62
J: Utilities											100

[a]See Düllmann and Masschelein (2007), p. 64

portfolios via Monte Carlo simulations with several parameter trials. This approach is related to Lopez (2004), who empirically determines the single correlation parameter for the ASRF model that leads to the same 99.9%-quantile as KMV's multi-factor model for several portfolio types (geographical region, PD, and asset size categories) using a grid search procedure. Thus, in the approach of Lopez (2004), the left-hand side of (5.7) is given and the single correlation parameter of the right-hand side is determined, whereas we are searching for the intra-sector correlation on the left-hand side that leads to a match of both models when the other parameters, especially the single correlation parameter of Basel II, are exogenously given.

As mentioned above, the required inter-sector correlation matrix could be estimated from historical default rates or from time series of stock returns.[275] Düllmann et al. (2008) demonstrate on the basis of an extensive simulation study that it is recommendable to use stock prices instead of historical default rates since this involves smaller statistical errors. Against this background, we rely on equity correlations, too, and use the correlation matrix of the MSCI EMU industry indices computed by Düllmann and Masschelein (2007) for the inter-sector correlation structure (see Table 5.1).[276]

Our definition of a well-diversified portfolio is based on the overall sector concentration of the German banking system, which can be found in Table 5.2.[277]

Even if it is theoretically possible to achieve lower capital requirements through a different sector decomposition, this can only be done by a restricted number of banks, since a deviation from the market structure of all banks immediately leads to a disequilibrium. In addition, the total number of credits is assumed to be n = 5,000 to guarantee low granularity.

[275]An overview of the literature regarding the measurement of asset correlation parameters can be found in Düllmann et al. (2008) and Grundke (2008).

[276]The correlation structure based on the MSCI US is similar, see Düllmann and Masschelein (2007).

[277]Düllmann and Masschelein (2007) notice that the concentration is very similar to other countries like France, Belgium, and Spain.

Table 5.2 Overall sector composition of the German banking system[a]

Sector	Exposure weight (%)
A: Energy	0.18
B: Materials	6.01
C1: Capital goods	11.53
C2: Comm. svs. and supplies	33.69
C3: Transportation	7.14
D: Consumer discretionary	14.97
E: Consumer staples	6.48
F: Health care	9.09
H: Information technology	3.20
I: Telecommunication services	1.04
J: Utilities	6.67

[a]Cf. Düllmann and Masschelein (2007), p. 63

Table 5.3 Implicit intra-sector correlations for different portfolio qualities

Portfolio type/quality	Implicit intra-sector correlation (%)
(I) Very high	30
(II) High	28
(III) Average	25
(IV) Low	23
(V) Very low	21

If we assume a constant intra-sector correlation, the best match is achieved by (approximately) $\rho_{\text{Intra}}^{(\text{Implied})} = 25\%$.[278] The concrete results, however, vary with the portfolio quality (see Table 5.3).[279] Thus, using a constant intra-sector correlation can lead to a significant underestimation of economic capital for high-quality portfolios and to an overestimation for low-quality portfolios.

To reduce the deviation, the intra-sector correlation should be decreasing in PD. We found that the following intra-sector correlation function leads to a good match for portfolios with different quality distributions:

$$\rho_{\text{Intra}}^{(\text{Implied})} = 0.185 \cdot \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}} + 0.34 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}}\right). \tag{5.8}$$

Thus, we use the correlation function type from Basel II but the correlation range is from 18.5 to 34% instead of 12 to 24%.[280] It has to be noted that this formula is

[278]This value results on the basis of both measures (VaR and ES) at the respective confidence level as described in Sect. 4.3.1. The result is consistent with Düllmann and Masschelein (2007), who use a constant intra-sector correlation of 25% in their analysis.

[279]See Fig. 4.7 for the portfolio characteristics.

[280]We tried several different functional forms but the formula above performed best. The multipliers 18.5% and 34% in function (5.8) were determined with a grid search using a reasonable parameter range, which is similar to the procedure of Lopez (2004) used for the single correlation parameter.

still a substantial simplification, as we assume that the intra-sector correlation is PD-dependent only. By contrast, empirically there are also inter-sectoral differences of this parameter.[281] In principle it would be possible to capture both effects, e.g. by multiplying a sector-specific factor to (5.8), which covers the relation of the empirically observed correlations.[282] Of course, the absolute level of the resulting correlations would usually be different from the empirical observations to keep Basel II consistent results. But for convenience, we rely on the PD-dependent formula (5.8) in our following analyses.

Hence, all additional input data needed for typical multi-factor models, e.g. using Monte Carlo simulations, are given with Table 5.1 and (5.8). Using these values, the multi-factor models should be consistent with the Basel framework. Thus, the measured economic capital is only lower than the regulatory capital if the portfolio is less concentrated than a typical, well-diversified portfolio, and the needed economic capital is above the capital requirement of the regulatory framework if there is more concentration risk in the credit portfolio. In order to avoid time-consuming Monte Carlo simulations, there exist some multi-factor models for an approximation of the portfolio risk. These will be presented subsequently.

5.2.2 *Accounting for Sector Concentrations with the Model of Pykhtin*

5.2.2.1 Derivation of the VaR-Based Multi-Factor Adjustment

In this section, the multi-factor adjustment of Pykhtin (2004) is examined. After explanation of the approach and derivation of the multi-factor adjustment formula for the VaR, the ES-based formula is calculated. Since the main shortcoming of the model is the time-consuming calculation for large portfolios, we focus on this issue thereafter and demonstrate how the approach can be implemented in a way that calculation time is reduced significantly.[283]

The multi-factor adjustment is an extension of the granularity adjustment presented in Chap. 4, which was introduced by Gordy (2003), Wilde (2001), and Martin and Wilde (2002), for multi-factor models and provides an analytical method for calculating the VaR and ES of a credit portfolio. The basic idea of Pykhtin is to approximate the portfolio loss $\tilde{L}$ in the multi-factor model with the respective portfolio loss $\tilde{\bar{L}}$ in an accurately adjusted ASRF model. This is done by

[281] E.g. Heitfield et al. (2006) determine the sector loadings, which equal $\sqrt{\rho_{\text{Intra}}}$, for 50 industry sectors using KMV data on asset values. The resulting intra-sector correlation is on average 18.8% and the standard deviation is 8.3%. These inter-sectoral differences are not captured by the formula above.

[282] A correlation structure with one degree of freedom for every PD/sector-combination is practically unfeasible due to high data requirements.

[283] In our setting, the computation time could be reduced by more than 99.8%.

mapping the correlation structure of all credits in the multi-factor model into a single correlation factor. This factor is determined by maximizing the correlation between the new single risk factor $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$. Based on this, a Taylor series expansion is performed around the constructed single-factor model.

Concretely, the distribution of $\tilde{\tilde{L}}$, which is the loss of the accurately adjusted single-factor model, can be calculated with the known formula of the ASRF model:[284]

$$\tilde{\tilde{L}} = \mu_{1,c}\left(\tilde{\tilde{x}}\right) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) - c_i \cdot \tilde{\tilde{x}}}{\sqrt{1-c_i{}^2}}\right], \tag{5.9}$$

where c_i is the correlation between the asset returns of two obligors, which is due to the conjoint dependence to the systematic risk factor $\tilde{\tilde{x}}$. Instead of using ρ as an input parameter as it is done in the ASRF model, the new correlation parameter c_i is calculated in a way that the correlation between the introduced single risk factor $\tilde{\tilde{x}}$ and the original sector factors $\{\tilde{x}_s\}$ is maximized. Thus, most of the correlation structure in the multi-factor model should be matched by this single factor.

As a next step, a Taylor series expansion around the comparable one-factor model (5.9) is performed in order to reduce the approximation error. Via this approach, it is possible to approximate the α-quantile $q_\alpha\left(\tilde{L}\right)$ of the portfolio loss by

$$q_\alpha\left(\tilde{L}\right) \approx q_\alpha\left(\tilde{\tilde{L}}\right) + \lambda \cdot \left[\frac{dq_\alpha\left(\tilde{\tilde{L}} + \lambda\tilde{Z}\right)}{d\lambda}\right]_{\lambda=0} + \frac{\lambda^2}{2} \cdot \left[\frac{d^2 q_\alpha\left(\tilde{\tilde{L}} + \lambda\tilde{Z}\right)}{d\lambda^2}\right]_{\lambda=0}, \tag{5.10}$$

where λ is the scale of perturbation and $\lambda\tilde{Z}$ describes the approximation error between "true" loss $\tilde{L}$ and the loss in the comparable one-factor model $\tilde{\tilde{L}}$, i.e. $\tilde{L} - \tilde{\tilde{L}} =: \lambda\tilde{Z}$. The first summand on the right-hand side of (5.10) is the α-quantile of the loss $\tilde{\tilde{L}}$ within the reasonably adjusted ASRF model, which is $\mu_{1,c}(\Phi^{-1}(1-\alpha))$.[285] The required correlation factor c_i is derived in Appendix 5.5.1.[286] In addition to maximizing the correlation between the single factor and the sector factors, the concrete choice of c_i guarantees that the first derivative in (5.10) is equal to zero, see also Appendix 5.5.1. Hence, the so-called multi-factor adjustment Δq_α is completely described by the second derivative in (5.10). According to Pykhtin (2004), the *multi-factor adjustment* Δq_α can be written as[287]

[284]The conditional PD stems from the Vasicek model, cf. Sect. 2.4 or 2.7.

[285]Cf. (5.9).

[286]For the determination of c_i, we need both the intra- and inter-sector correlations, which can be taken from Sect. 5.2.1.

[287]This formula has already been derived for the granularity adjustment formula, cf. (4.18).

$$\Delta q_\alpha = q_\alpha(\tilde{L}) - q_\alpha\left(\tilde{\tilde{L}}\right)$$
$$\approx -\frac{1}{2 \cdot d\mu_{1,c}(\bar{x})/d\bar{x}} \cdot \left[\frac{d\eta_{2,c}(\bar{x})}{d\bar{x}} - \eta_{2,c}(\bar{x}) \cdot \left(\frac{d^2\mu_{1,c}(\bar{x})/d\bar{x}^2}{d\mu_{1,c}(\bar{x})/d\bar{x}} + \bar{x}\right)\right]\Bigg|_{\bar{x}=\Phi^{-1}(1-\alpha)}, \tag{5.11}$$

where $\eta_{m,c}(\bar{x}) := \eta_m(\tilde{L}|\tilde{\bar{x}} = \bar{x})$ is the mth conditional moment of the portfolio loss about the mean.

The conditional expectation $\mu_{1,c}(\bar{x})$ and the required derivatives are already known from the granularity adjustment:[288]

$$\mu_{1,c}(\bar{x}) = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot p_i(\bar{x}), \tag{5.12}$$

$$\frac{d\mu_{1,c}(\bar{x})}{d\bar{x}} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d(p_i(\bar{x}))}{d\bar{x}}, \tag{5.13}$$

$$\frac{d^2\mu_{1,c}(\bar{x})}{d\bar{x}^2} = \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \frac{d^2(p_i(\bar{x}))}{d\bar{x}^2}, \tag{5.14}$$

with

$$p_i(\bar{x}) = \Phi\left(\frac{\Phi^{-1}(PD_i) - c_i \cdot \bar{x}}{\sqrt{1 - c_i{}^2}}\right), \tag{5.15}$$

$$\frac{d(p_i(\bar{x}))}{d\bar{x}} = -\frac{c_i}{\sqrt{1 - c_i{}^2}} \cdot \varphi\left(\frac{\Phi^{-1}(PD_i) - c_i \cdot \bar{x}}{\sqrt{1 - c_i{}^2}}\right), \tag{5.16}$$

and

$$\frac{d^2(p_i(\bar{x}))}{d\bar{x}^2} = -\frac{c_i{}^2}{1 - c_i{}^2} \cdot \frac{\Phi^{-1}(PD_i) - c_i \cdot \bar{x}}{\sqrt{1 - c_i{}^2}} \cdot \varphi\left(\frac{\Phi^{-1}(PD_i) - c_i \cdot \bar{x}}{\sqrt{1 - c_i{}^2}}\right). \tag{(5.17)}$$

The conditional variance $\eta_{2,c}$ is

$$\eta_{2,c} = \mathbb{V}\left(\sum_{i=1}^{n} w_i \cdot \widetilde{LGD}_i \cdot 1_{\{\tilde{D}_i\}}|\tilde{\bar{x}} = \bar{x}\right) \tag{5.18}$$

[288] Cf. Sect. 4.2.1.2.

but in contrast to the single risk-factor framework, the defaults are not independent conditional on $\bar{x}$. Thus, it is not possible to use the formula of the granularity adjustment. The dependence structure of the conditional default events becomes apparent if we rewrite the formula of the asset return (5.1) using (5.2) and (5.73):

$$
\begin{aligned}
\tilde{a}_{s,i} &= \sqrt{\rho_{\text{Intra},i}} \cdot \tilde{x}_s + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} - c_i \cdot \tilde{\bar{x}} + \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} - c_i \cdot \sum_{k=1}^{K} b_k \cdot \tilde{z}_k + \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i \\
&= c_i \cdot \tilde{\bar{x}} + \sum_{k=1}^{K} \left(\sqrt{\rho_{\text{Intra},i}} \cdot \alpha_{s,k} - c_i \cdot b_k \right) \cdot \tilde{z}_k + \sqrt{1-\rho_{\text{Intra},i}} \cdot \tilde{\xi}_i .
\end{aligned}
\tag{5.19}
$$

Even if the systematic factor $\tilde{\bar{x}}$ is fixed, the asset returns are not independent of each other but depend on the constructed sector variables $\tilde{z}_k$.[289] The correlation between obligor i and j conditional on $\tilde{\bar{x}}$ can be calculated as:[290]

$$
\begin{aligned}
\rho_{ij}^{\bar{x}} &= \text{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j} | \tilde{\bar{x}}\right) \\
&= \frac{\sqrt{\rho_{\text{Intra},i} \cdot \rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - c_i \cdot c_j}{\sqrt{\left(1-c_i{}^2\right) \cdot \left(1-c_j{}^2\right)}} .
\end{aligned}
\tag{5.20}
$$

Although the asset returns are not independent conditional on $\tilde{\bar{x}}$, they are independent conditional on the sector factors $\tilde{z}_k$. We can use this property by decomposing the conditional variance of the portfolio loss $\eta_{2,c}(\bar{x})$ into two terms, $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\text{GA}}(\bar{x})$:[291]

$$
\eta_{2,c}(\bar{x}) = \mathbb{V}\left(\tilde{L} | \tilde{\bar{x}} = \bar{x}\right) = \underbrace{\mathbb{V}\left[\mathbb{E}\left(\tilde{L} | \{\tilde{z}_k\}\right) | \tilde{\bar{x}} = \bar{x}\right]}_{\eta_{2,c}^{\infty}(\bar{x})} + \underbrace{\mathbb{E}\left[\mathbb{V}\left(\tilde{L} | \{\tilde{z}_k\}\right) | \tilde{\bar{x}} = \bar{x}\right]}_{\eta_{2,c}^{\text{GA}}(\bar{x})} . \tag{5.21}
$$

The term $\eta_{2,c}^{\infty}(\bar{x})$ describes the systematic risk adjustment, which is given by the difference between the multi-factor and single-factor loss distribution in infinitely

[289] Cf. (5.2).

[290] See Appendix 5.5.2.

[291] The derivation of the variance decomposition can be found in Weiss (2005), p. 385 f.

fine-grained portfolios. The other term $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ is relevant for the granularity adjustment, which measures the influence of portfolio name concentration. The calculation of the terms $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ can be found in Appendix 5.5.3 and utilizes the conditional independence property of the decomposed terms. This leads to

$$\eta_{2,c}^{\infty}(\bar{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \left[\Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}\left(p_j(\bar{x})\right), \rho_{ij}^{\bar{x}}\right) - p_i(\bar{x})p_j(\bar{x})\right], \tag{5.22}$$

$$\eta_{2,c}^{\mathrm{GA}}(\bar{x}) = \sum_{i=1}^{n} w_i^{\,2}\left(ELGD_i^{\,2}\left[p_i(\bar{x}) - \Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}(p_i(\bar{x})), \rho_{ii}^{\bar{x}}\right)\right] + VLGD_i p_i(\bar{x})\right). \tag{5.23}$$

According to (5.11), we also need the derivative $d\eta_{2,c}(\bar{x})/d\bar{x}$. Thus, the derivatives of the decomposed variance terms are calculated in Appendix 5.5.4, leading to

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\bar{x})}{d\bar{x}} &= 2\cdot\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \frac{dp_i(\bar{x})}{d\bar{x}} \\ &\quad\cdot\left(\Phi\left(\frac{\Phi^{-1}\left[p_j(\bar{x})\right] - \rho_{ij}^{\bar{x}}\Phi^{-1}\left[p_i(\bar{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) - p_j(\bar{x})\right),\end{aligned} \tag{5.24}$$

$$\begin{aligned}\frac{d\eta_{2,c}^{\mathrm{GA}}(\bar{x})}{d\bar{x}} &= \sum_{i=1}^{n} w_i^2 \frac{dp_i(\bar{x})}{d\bar{x}}\cdot\left(ELGD_i^2\left[1 - 2\Phi\left(\frac{\Phi^{-1}[p_i(\bar{x})] - \rho_{ii}^{\bar{x}}\Phi^{-1}[p_i(\bar{x})]}{\sqrt{1-\left(\rho_{ii}^{\bar{x}}\right)^2}}\right)\right]\right. \\ &\quad\left. + VLGD_i\right).\end{aligned} \tag{5.25}$$

Using the terms (5.13)–(5.17), (5.20), and (5.22)–(5.25), the multi-factor adjustment (5.11) can be calculated. Since the multi-factor adjustment is linear in the conditional variance and its derivatives, we can also write the multi-factor adjustment as

$$\Delta q_\alpha = \Delta q_\alpha^{\infty} + \Delta q_\alpha^{\mathrm{GA}}, \tag{5.26}$$

i.e. the multi-factor adjustment can be split into a systematic risk adjustment component and a granularity adjustment component. To sum up, the approximation of a loss quantile $q_\alpha(\tilde{L})$ in (5.10) is given by (5.9) and by the multi-factor adjustment

$$q_\alpha(\tilde{L}) \approx q_\alpha\left(\bar{\tilde{L}}\right) + \Delta q_\alpha = q_\alpha\left(\bar{\tilde{L}}\right) + \Delta q_\alpha^{\infty} + \Delta q_\alpha^{\mathrm{GA}}. \tag{5.27}$$

5.2.2.2 Derivation and Implementation of the ES-Based Multi-Factor Adjustment

After dealing with the VaR, now the ES-based multi-factor adjustment is presented. Using the integral representation of the ES (2.20) and substituting the quantile $q_\alpha(\tilde{L})$ by approximation (5.27), the ES can be written as

$$\begin{aligned} ES_\alpha(\tilde{L}) &= \frac{1}{1-\alpha} \cdot \int_\alpha^1 q_s(\tilde{L})ds \\ &\approx \frac{1}{1-\alpha} \cdot \int_\alpha^1 \left(q_s\left(\tilde{\bar{L}}\right) + \Delta q_s\right)ds \\ &= ES_\alpha\left(\tilde{\bar{L}}\right) + \frac{1}{1-\alpha} \cdot \int_\alpha^1 \Delta q_s ds =: ES_\alpha\left(\tilde{\bar{L}}\right) + \Delta ES_\alpha. \end{aligned} \tag{5.28}$$

The first summand of the right-hand side describes the ES for the comparable single-factor model and the second summand is the multi-factor adjustment.

The ES in the ASRF model is already known from (4.59), leading to

$$ES_\alpha\left(\tilde{\bar{L}}\right) = \frac{1}{1-\alpha} \sum_{i=1}^{n} w_i \cdot ELGD_i \cdot \Phi_2\left(-\Phi^{-1}(\alpha), \Phi^{-1}(PD_i), c_i\right). \tag{5.29}$$

In order to calculate the multi-factor adjustment in (5.28), we use the formulation of Δq_s from (4.18):

$$\Delta ES_\alpha(\tilde{L}) = -\frac{1}{2(1-\alpha)} \int_\alpha^1 \frac{1}{\varphi(\bar{x})} \frac{d}{d\bar{x}} \left(\frac{\varphi(\bar{x})\, \eta_{2,c}(\bar{x})}{d\mu_{1,c}(\bar{x})/d\bar{x}} \right)\Bigg|_{\bar{x}=\Phi^{-1}(1-s)} ds. \tag{5.30}$$

Substituting $x := \Phi^{-1}(1-s)$ and thus $ds = -\varphi(x)dx$, $x(s=\alpha) = \Phi^{-1}(1-\alpha)$, and $x(s=1) = -\infty$ results in

$$\begin{aligned} \Delta ES_\alpha(\tilde{L}) &= -\frac{1}{2(1-\alpha)} \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \frac{1}{\varphi(x)} \frac{d}{d\bar{x}} \left(\frac{\varphi(\bar{x})\, \eta_{2,c}(\bar{x})}{d\mu_{1,c}(\bar{x})/d\bar{x}} \right)\Bigg|_{\bar{x}=x} \varphi(x)dx \\ &= -\frac{1}{2(1-\alpha)} \left[\left(\frac{\varphi(x)\, \eta_{2,c}(x)}{d\mu_{1,c}(x)/dx} \right) \right]_{-\infty}^{\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{5.31}$$

The derivative of $\mu_{1,c}$ can be written as $d\mu_{1,c}(x)/dx = g \cdot \varphi(x)$, with g being a constant value. As $\eta_{2,c}(-\infty) = 0$, the right-hand side of (5.31) vanishes at $x = -\infty$, leading to

$$\Delta ES_\alpha(\tilde{L}) = -\frac{1}{2(1-\alpha)} \frac{\varphi(x)\,\eta_{2,c}(x)}{d\mu_{1,c}(x)/dx}\bigg|_{x=\Phi^{-1}(1-\alpha)}. \tag{5.32}$$

This equation can easily be computed using the conditional variance and the derivative of the conditional expectation of Sect. 5.2.2.1. Again, the multi-factor adjustment can be decomposed into a systematic and an idiosyncratic part by decomposing the conditional variance. Hence, the ES for a portfolio in a multi-factor model is given by

$$ES_\alpha(\tilde{L}) = ES_\alpha\left(\tilde{\bar{L}}\right) + \Delta ES_\alpha^\infty + \Delta ES_\alpha^{\mathrm{GA}}. \tag{5.33}$$

It is worth noticing that the resulting expression (5.32) is much simpler than the corresponding formula for the VaR. The same phenomenon could already be observed for the granularity adjustment formula in Chap. 4.

In principle, it is straightforward to implement the Pykhtin model. For calculating the ES we have to compute (5.32). The problem is that the computation can be extremely time-consuming if the formula is applied to large portfolios. The reason is that the calculation procedure inter alia requires n^2-times the computation of the conditional asset correlation,[292] with n being the number of credits. An alternative performed by Düllmann and Masschelein (2007) is to neglect the multi-factor adjustment and to use (5.9) only to aggregate all credits for each sector and thus using the formulas on sector and not on borrower level. Of course, it may be expected that this simplification is at the cost of lower approximation accuracy. To consider the multi-factor adjustment, we propose to build PD-classes for each of the sectors and aggregate the credits to these buckets for the calculation of the multi-factor adjustment, so that the computation time is predominated by

$$\text{Loops} = (N_{\mathrm{PD}} \cdot S)^2, \tag{5.34}$$

where N_{PD} and S denote the number of PD-classes and sectors, respectively.[293] If the number of PD-classes is sufficient, the approximation error resulting from aggregating individual PDs to PD-classes is negligible. As the number of loops does not grow with bigger portfolios, it is possible to perform the adjustment on

[292] The quadratic computation effort is due to the determination of a double sum (see (5.22) and (5.24)).

[293] The results of the multi-factor adjustment do not differ whether different exposures with the same PD are aggregated or handled separately on borrower level. For details see Sect. 5.2.2.1 and Appendix 5.5.1.

bucket level within reasonable time. Only the granularity adjustment should be calculated on borrower level but this is no computational burden.[294]

5.2.3 *Accounting for Sector Concentrations with the Model of Cespedes, Herrero, Kreinin and Rosen*

5.2.3.1 Design of the Diversification Factor

Cespedes et al. (2006) present a method to relate the economic capital in the multi-factor model to the regulatory capital formula.[295] These models are linked via a *diversification factor* $DF(\cdot)$, which depends on two parameters:

- The average sector concentration HHI and
- The average weighted inter-sector correlation $\bar{\beta}$

Herewith, the economic capital of a portfolio can be approximated as:

$$EC^{\mathrm{mf}} \approx DF\left(HHI, \overline{\beta}\right) \cdot RC. \tag{5.35}$$

Thus, the economic capital in the multi-factor model EC^{mf} can be approximated by a well-defined diversification factor DF multiplied with the regulatory capital requirement of the ASRF model RC. As mentioned before, Cespedes et al. (2006) assume in their calculations the regulatory capital of Pillar 1 to be an upper barrier of the true risk because no diversification effects between the sectors are considered, which in turn implies the parameter DF to be always less than or equal to one. In contrast, if we use our definition of the intra-sector correlation ρ_{Intra} from Sect. 5.2.1, it is possible to obtain $EC^{\mathrm{mf}} > RC$ as well as $EC^{\mathrm{mf}} < RC$ depending on the degree of diversification in comparison to the well-diversified portfolio defined in Sect. 5.2.1. Hence, our later on calculated DF-function can be greater than one, i.e. the DF-function measures not only the benefit from sector diversification but also the risk resulting from high sector concentration. As the regulatory capital is additive in the ASRF model, (5.35) can be substituted by

$$EC^{\mathrm{mf}} \approx DF \cdot \sum_{s=1}^{S} RC^{s}, \tag{5.36}$$

in which EC^{mf} is the economic capital in the multi-factor model and RC^{s} is the regulatory capital for sector s. In principle, the approach can be characterized as

---

[294]The computation time when calculating the multi-factor adjustment on bucket- instead on borrower-level can be reduced from 67 min to 5 s for a portfolio with 11 sectors, 7 PD-classes, and 5,000 creditors.

[295]In the strict sense, Cespedes et al. (2006) relate the multi-factor model to the *economic* capital in a single-factor model. But since they apply the regulatory capital formula and we require a relation to this formula, too, we use the term regulatory capital instead.

follows: Firstly, EC^{mf} is calculated for a multitude of portfolios via Monte Carlo simulations. For each simulated portfolio, the diversification factor can be calculated according to (5.36). Finally, a regression is performed to get an approximation for DF as a function of the two parameters HHI and $\overline{\beta}$. If DF can capture the industry diversification effects, we are able to approximate EC^{mf} with (5.36) without additional Monte Carlo simulations.

To derive the parameters which explain the effect of diversification and concentration in a multi-factor model, Cespedes et al. (2006) suggest to use the average inter-sector correlation $\overline{\beta}$. This can be interpreted as a scale of the dependence between the sectors. The formula for $\overline{\beta}$ is given as

$$\overline{\beta} = \frac{\sum_{s=1}^{S} \sum_{t \neq s} \rho_{s,t}^{\mathrm{Inter}} \cdot RC^s \cdot RC^t}{\sum_{s=1}^{S} \sum_{t \neq s} RC^s \cdot RC^t}, \tag{5.37}$$

The correlation is weighted by the regulatory capital in order to account for the contribution of each sector. As a consequence, the correlation between sectors with a high capital requirement account for a high degree of the average correlation.[296] The second suggested parameter is a parameter for the degree of capital diversification, measured by the Herfindahl–Hirschmann Index HHI.[297] It describes the sector concentration measured by the relative weight of each sectors regulatory capital RC^s:[298]

$$HHI = \frac{\sum_{s=1}^{S} (RC^s)^2}{\left(\sum_{s=1}^{S} RC^s \right)^2}. \tag{5.38}$$

As mentioned in Sect. 3.4, the parameter HHI lies between two extreme values:

- $HHI = 1/S$, i.e. perfect sector diversification,
- $HHI = 1$, i.e. perfect sector concentration.

To avoid a too complex model, Cespedes et al. (2006) neglect further potential input parameters to determine the DF-function. To approximate the multi-factor model, (5.36) can be rewritten as

$$EC^{\mathrm{mf}} \approx DF\left(HHI, \overline{\beta}\right) \cdot \sum_{s=1}^{S} RC^s. \tag{5.39}$$

[296]The idea is related to Pykhtin (2004), who uses the VaR from the ASRF model as a weight when maximizing the correlation between the single factor of the comparable one-factor model and the sector factors; cf. (5.82)–(5.85).

[297]Cespedes et al. (2006) call this parameter the capital diversification index (CDI).

[298]This concentration measure corresponds to (2.87).

5.2.3.2 Computation of the Diversification Factor by Simulation

In the following, our procedure to estimate the DF-function is presented. In order to get a universally valid DF-factor, as many portfolios as possible have to be generated and simulated. To reduce the necessary number of trials, the portfolios should be restricted to those with reasonable characteristics. Our portfolios are randomly generated using the following parameter setting. When we state several parameter values or a parameter range, the parameter is randomly drawn from this set.

For the intra-sector correlations, we use the functional form of (5.8). The inter-sector correlation structure is taken from Table 5.1, so that all simulated portfolios are stemming from this sector definition. Each portfolio consists of {2, ..., 11} sectors that are randomly drawn from the different industries. The sector weights are in [0, 1] and sum up to one. The total number of credits is 5,000, equally divided for each sector. Each sector in turn consists of credits from the PD classes {AAA, AA, A, BBB, BB, B, CCC}. Instead of using equally distributed PD classes, we draw the quality distribution from our predefined credit portfolio qualities {very high, high, average, low, very low} for every sector from Fig. 4.7.[299] We draw 25,000 or 50,000 portfolios and compute the economic capital in the multi-factor model for each portfolio.

To determine the economic capital, we have tried both Monte Carlo simulations with 100,000 trials[300] for every portfolio and the Pykhtin formula from Sect. 5.2.2. Because the computation time for Monte Carlo simulations is materially longer, the corresponding results are based on 25,000 random portfolios, whereas we computed the economic capital for 50,000 portfolios when using the Pykhtin formula instead. Furthermore, since Cespedes et al. (2006) use the VaR as the relevant risk measure and thus define the economic capital as $EC^{\mathrm{mf}} := VaR^{\mathrm{mf}}_{0.999} - EL$, we have to redefine the economic capital of the multi-factor model with respect to ES as argued in Sect. 4.3.1: $EC^{\mathrm{mf}} := ES^{\mathrm{mf}}_{0.9972} - EL$.[301] In contrast, for the regulatory capital we use $RC = VaR^{(\mathrm{Basel})} - EL$. The result could also be related to the Expected Shortfall in the ASRF model but we have detected that the results differ only marginally and the VaR is easier to implement in typical spreadsheet applications.[302] The results for the diversification factor DF are very similar regardless of whether they are based on

[299] The setting is similar to Cespedes et al. (2006). Until this point, the main difference is the definition of the intra- and inter-sector correlations.

[300] For the determination of the economic capital for one specific portfolio, the number of trials is slightly low but as we perform 25,000 simulations and the simulation noise of each simulation is unsystematic, the error terms should cancel out each other to a large extent.

[301] We have also tested the results when using the ES instead of the unexpected loss but the coefficient of determination is higher when subtracting the EL in the corresponding formulas when performing the simulations.

[302] To determine the Expected Shortfall with (4.59), a bivariate cumulative normal distribution has to be computed whereas the Value at Risk only makes use of univariate distributions.

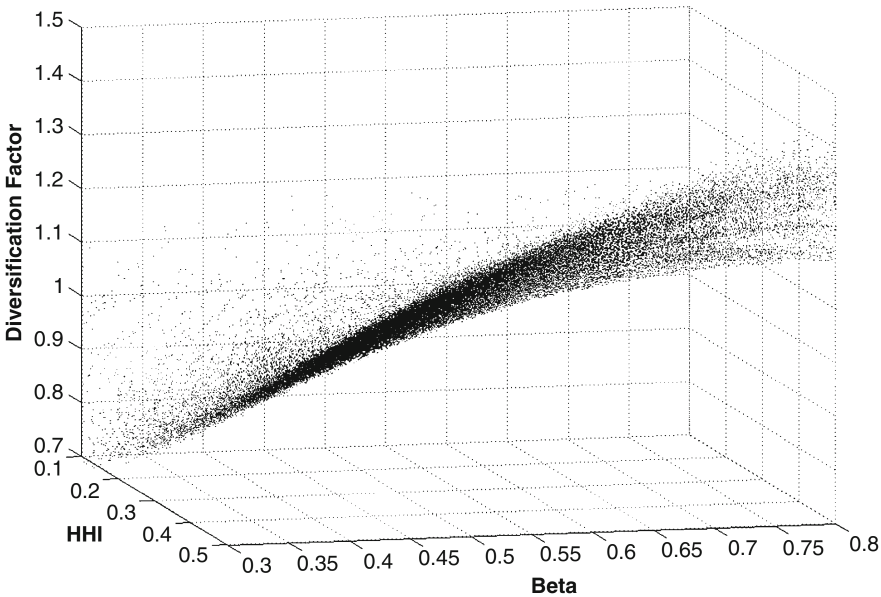

Fig. 5.1 Diversification Factor realizations on the basis of 50,000 simulations

Monte Carlo simulations or on the Pykhtin formula. Fig. 5.1 presents characteristics of the diversification factor when using the Pykhtin formula.

For a determination of the functional form of DF, we use a regression of the type[303]

$$\begin{aligned} DF = a_0 + a_1 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) \\ + a_2 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + a_3 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \end{aligned} \tag{5.40}$$

in both cases, using the ordinary least squares (OLS) technique. The resulting function when using Monte Carlo simulations is

$$\begin{aligned} DF_{\mathrm{MC}} = 1.4626 - 1.4475 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) \\ - 0.0382 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + 0.3289 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \end{aligned} \tag{5.41}$$

[303] We have tried several different regressions but similar to Cespedes et al. (2006), this function worked best. In contrast to Cespedes et al. (2006) we do not set the first parameter a_0 to one because our DF-factor is not bounded by the single-factor model.

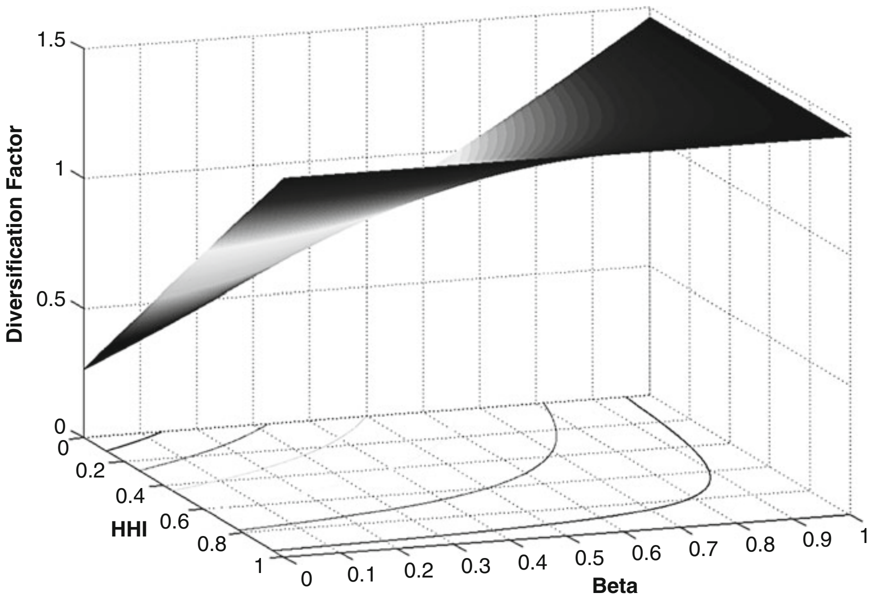

Fig. 5.2 Surface plot of the *DF*-function

with $R^2 = 95.5\%$. Analogously, we determined the *DF*-function when using the Pykhtin formula

$$\begin{aligned} DF_{\text{Pykhtin}} = {} & 1.4598 - 1.4168 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right) \\ & - 0.0213 \cdot (1 - HHI)^2 \cdot \left(1 - \overline{\beta}\right) + 0.2421 \cdot (1 - HHI) \cdot \left(1 - \overline{\beta}\right)^2 \end{aligned} \tag{5.42}$$

with a coefficient of determination of $R^2 = 97.9\%$. The latter function is plotted in Fig. 5.2.[304] To finally get the approximation for the multi-factor model, (5.39) has to be computed using either function (5.41) or (5.42).

It can be seen that the maximum diversification factor is about 1.46. Thus, in the case of (almost) no diversification effects, the measured capital requirement is 46% above the regulatory capital under Pillar 1. This will appear in the case of being concentrated to a single sector, leading to $HHI = 1$, as well as in the theoretical case

[304] The shape of the function is similar to Cespedes et al. (2006) but their range is from 0.1 to 1.0 whereas our function ranges from 0.2 to 1.5. In addition, they received a little higher R^2 (99.4% instead of 95.5% or 97.9%) but this is mainly due to the different simulation setting. Cespedes et al. (2006) directly draw the parameter $\overline{\beta}$ as an input parameter for each simulation, implying $\overline{\beta}$ to fully define their correlation structure. We use a heterogeneous correlation structure instead and compute $\overline{\beta}$ for the portfolios. Thus, in our setting $\overline{\beta}$ does not reflect the complete correlation structure, which results in a lower R^2 but does not imply a worse approximation.

of perfect correlations between the relevant sectors, leading to $\bar{\beta} = 1$. Furthermore, the diversification factor is strongly increasing in *HHI* and in $\bar{\beta}$, which is consistent with the intuition.

5.2.4 Accounting for Sector Concentrations with the Model of Düllmann

5.2.4.1 The Binomial Expansion Technique

The model of Düllmann (2006) is a combination of the Binomial Expansion Technique (BET)-model and the Infection Model of Davis and Lo (2001). For this reason, at first, the BET-model and the infection model will be explained, before the model of Düllmann will be presented and applied to our multi-factor setting. During the application, we will deviate from the original procedure in order to apply the ES instead of the VaR and to accelerate the computation time significantly for large portfolios.[305]

The *Binomial Expansion Technique* (BET) was developed by Moody's for the rating of CDOs but it can also be applied to standard credit portfolios without tranches. The BET-model approximates the loss distribution of the portfolio but is much less computationally intensive than Monte Carlo simulations.[306] The main idea is to perform a mapping of the original portfolio into a hypothetical homogeneous portfolio with stochastically independent, Bernoulli distributed loss events leading to a binomial distributed number of losses. The hypothetical portfolio can be described by the average probability of default $\overline{p}$, the number of credits D, which is called the modified Diversity Score, and the (constant) Loss Given Default LGD. The parameters D and $\overline{p}$ are calibrated in a way that the first two moments of the original and the hypothetical portfolio loss distribution are identical. This shall lead to a similar overall loss distribution of both portfolios.

With n_s for the number of credits in sector s, the loss of the original portfolio equals

$$\tilde{L}^{\text{orig}} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot 1_{\{\tilde{D}_{s,i}\}}, \tag{5.43}$$

whereas the loss of the hypothetical portfolio is

$$\tilde{L}^{\text{hyp}} = \sum_{i=1}^{D} w \cdot LGD \cdot 1_{\{\tilde{D}_i\}} = \sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot 1_{\{\tilde{D}_i\}}. \tag{5.44}$$

[305]In comparison to the original procedure, the computation time could be reduced by almost 99.9% in our calculations.

[306]Cf. Cifuentes et al. (1996), Cifuentes and O'Connor (1996), and Cifuentes and Wilcox (1998).

Matching the expectation for both portfolios leads to[307]

$$\overline{p} := \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i} \cdot PD_{s,i} \tag{5.45}$$

and matching the variance results in[308]

$$D = \frac{\overline{p}\cdot(1-\overline{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.46}$$

The pairwise default correlation and the asset correlation between borrower i in sector s and borrower j in sector t can be transformed into each other with[309]

$$\mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right) = \frac{\Phi_2\left(\Phi^{-1}(PD_i),\Phi^{-1}\left(PD_j\right),\mathrm{Corr}\left(\tilde{a}_{s,i},\tilde{a}_{t,j}\right)\right) - PD_i\cdot PD_j}{\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.47}$$

In the original model, it is assumed that the correlation between every two borrowers, which are in the same sector, is identical. Furthermore, it is assumed that the correlation between two borrowers in distinct sectors is always identical and the PDs inside a sector are homogeneous. These assumptions would lead to some simplifications in (5.45)–(5.47), but they are not necessary for the calculation of the loss distribution. Thus, we can also use the correlation structure from (5.4) and use (5.45)–(5.47). Having determined the parameters $\overline{p}$ and D, we can calculate the loss distribution for the hypothetical portfolio. Since the (uncertain) number of defaults $\tilde{k}$ in the hypothetical portfolio is binomially distributed

$$\tilde{k} = \sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} \sim \mathcal{B}(D,\overline{p}), \tag{5.48}$$

the probability of having k defaults is

$$P_k = \mathbb{P}\left(\tilde{k}=k\right) = \mathbb{P}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} = k\right) = \binom{D}{k}\cdot(\overline{p})^k\cdot(1-\overline{p})^{D-k}. \tag{5.49}$$

[307]See Appendix 5.5.5.

[308]See Appendix 5.5.5.

[309]See Appendix 5.5.6.

The corresponding cumulative distribution function for the number of defaults is

$$F_k(x) = \mathbb{P}\left(\tilde{k} \leq x\right) = \mathbb{P}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}} \leq x\right) = \sum_{k=0}^{x} P_k. \tag{5.50}$$

Thus, the loss distribution of the original portfolio can be approximated with

$$\begin{aligned} F_{\text{orig}}^{(n)}(l) \approx F_{\text{hyp}}^{(D)}(l) &= \mathbb{P}\left(\sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot 1_{\{\tilde{D}_i\}} \leq l\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}} \leq \frac{l \cdot D}{LGD}\right) = \sum_{k=0}^{\lfloor l \cdot D/LGD \rfloor} P_k, \end{aligned} \tag{5.51}$$

leading to a VaR of

$$VaR_\alpha\left(\tilde{L}^{\text{orig}}\right) \approx VaR_\alpha\left(\tilde{L}^{\text{hyp}}\right) = \frac{1}{D} \cdot LGD \cdot F_k^{-1}(\alpha), \tag{5.52}$$

where $F_k^{-1}(\alpha)$ is the inverse CDF of the binomial distribution with parameters D and $\overline{p}$ from (5.48). The ES can be computed using the definition of the ES (2.19). From (5.48) and (5.52) it can best be seen that the interaction between the credits is incorporated by reducing the real number of credits to the hypothetical number, the Diversity Score, with higher exposure weights. E.g., for $D = n/2$, each (stochastically independent) default in the hypothetical portfolio is equivalent to two defaults in the original portfolio, which leads to some kind of default interaction in the original portfolio.

5.2.4.2 The Infectious Defaults Model

Davis and Lo (2001) present an alternative to the BET-model for the determination of the loss distribution of a credit portfolio which is assumed to be homogeneous.[310] In the model, credits can default not only directly but they can also be "*infected*" by other credits leading to an indirect default. Similar to the BET-model, the direct defaults are assumed to be stochastically independent, leading to a binomial distribution of direct defaults. Thus, the task is how the indirect defaults can be incorporated into the loss distribution. To begin with, several indicator variables are introduced, which indicate the type of default and the interaction. Whether a credit defaults or not is expressed by the indicator variable $\tilde{Z}_i$, which equals one in the event of default and zero otherwise. Thus, the total number of defaults in the portfolios is

$$\tilde{k} = \tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n. \tag{5.53}$$

[310]Similar to the BET-model, the authors developed their model for CDOs but it can also be applied to standard credit portfolios.

If credit i defaults directly, the indicator variable $\tilde{X}_i$ takes the value one. Furthermore, the indicator variable $\tilde{Y}_{j,i}$ indicates whether credit j could *potentially* infect credit i. The condition for this infection is that both the infection variable $\tilde{Y}_{j,i}$ and the direct default indicator $\tilde{X}_j$ of credit j take the value one. This leads to the following function for the default indicator $\tilde{Z}_i$:

$$\tilde{Z}_i = \tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left(1 - \prod_{j \neq i} \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)\right), \text{with } i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, n. \tag{5.54}$$

In (5.54), the second term is only relevant if credit i does not default directly. In this case, an infection through any one or several credits leads to a product of zero so that the second term equals one. The equation will be demonstrated further with the following examples for a portfolio consisting of four credits:

- Credit 1 defaults directly:

$$\begin{aligned} Z_1 &= X_1 + (1 - X_1) \cdot \left(1 - \prod_{j \neq i} \left(1 - X_j \cdot Y_{j,i}\right)\right) \\ &= 1 + (1 - 1) \cdot \left(1 - \prod_{j \neq i} \left(1 - X_j \cdot Y_{j,i}\right)\right) = 1. \end{aligned}$$

  As the term $(1 - X_1)$ equals zero, the last expression vanishes and Credit 1 defaults directly without an effect of defaults of the other credits.

- Credit 2 defaults as a consequence of infection from the defaulted credit 1:

$$\begin{aligned} Z_2 &= X_2 + (1 - X_2) \cdot \left(1 - \left(1 - X_1 \cdot Y_{1,2}\right) \cdot \left(1 - X_3 \cdot Y_{3,2}\right) \cdot \left(1 - X_4 \cdot Y_{4,2}\right)\right) \\ &= 0 + (1 - 0) \cdot (1 - (1 - 1 \cdot 1) \cdot (1 - 0 \cdot 1) \cdot (1 - 1 \cdot 0)) = 1. \end{aligned}$$

  The non-defaulting Credit 3 would also have the potential to infect Credit 2 in the case of a default. Credit 4 defaults but does not infect credit 2.

- Credit 3 does not default:

$$\begin{aligned} Z_3 &= X_3 + (1 - X_3) \cdot \left(1 - \left(1 - X_1 \cdot Y_{1,3}\right) \cdot \left(1 - X_2 \cdot Y_{2,3}\right) \cdot \left(1 - X_4 \cdot Y_{4,3}\right)\right) \\ &= 0 + (1 - 0) \cdot (1 - (1 - 1 \cdot 0) \cdot (1 - 0 \cdot 0) \cdot (1 - 1 \cdot 0)) = 1. \end{aligned}$$

  The third credit does neither default directly nor indirectly.

  In a probabilistic setting, a direct default is assumed to happen with probability p:

$$\mathbb{P}\left(\tilde{X}_i = 1\right) = p \quad \forall i. \tag{5.55}$$

  Similar, the infection indicator $\tilde{Y}_{j,i}$ takes the value one with probability q:

$$\mathbb{P}\left(\tilde{Y}_{j,i} = 1\right) = q \quad \forall i, j. \tag{5.56}$$

Thus, the dependence structure is assumed to be perfectly homogeneous. Let i be the number of direct defaults, $k–i$ the number of indirect defaults, so that we have in total k defaults, and the other $n–k$ credits do not default. The probability of observing k defaults out of n credits is

$$P_k = \binom{n}{k} \cdot \sum_{i=1}^{k} \binom{k}{i} \cdot \underbrace{p^i}_{i \text{ direct defaults}} \cdot \underbrace{\left[(1-p) \cdot \left(1-(1-q)^i\right)\right]^{k-i}}_{k-i \text{ indirect defaults}} \cdot \underbrace{\left[(1-p) \cdot (1-q)^i\right]^{n-k}}_{n-k \text{ survivors}} . \tag{5.57}$$

The probability P_k can be split into four parts:

(a) If we ignore the perturbations, the probability of i direct defaults is p^i.
(b) A number of $k–i$ indirect defaults occurs if these credits do not default directly, which has the probability $(1-p)^{k-i}$, but these are infected by any of the i directly defaulted credits with probability $(1-(1-q)^i)^{k-i}$.
(c) For a survival of $n–k$ credits, these credits default neither directly, which has a probability of $(1-p)^{n-k}$, nor any of the i directly defaulted credits leads to an indirect default, which can be expressed as $((1-q)^i)^{n-k}$.
(d) There are several possible perturbations of defaulted credits. Firstly, there are $\binom{n}{k}$ perturbations for k out of n defaults. Furthermore, there are several combinations of direct and indirect defaults. A number of k defaults can consist of $(1;\ k-1)$, $(2;\ k-2)$, ..., $(k;\ 0)$ direct and indirect defaults. For each of these combinations, there exist $\binom{k}{i}$ perturbations. All of the corresponding probabilities have to be summed up to cover all combinations for k defaults.

Expression (5.57) could also be written as

$$\begin{aligned}
P_k &= \binom{n}{k} \cdot \sum_{i=1}^{k} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)} \\
&= \binom{n}{k} \cdot \sum_{i=1}^{k-1} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)} \\
&\quad + \binom{n}{k} \cdot \binom{k}{k} \cdot p^k \cdot (1-p)^{n-k} \cdot \left(1-(1-q)^k\right)^0 \cdot (1-q)^{k(n-k)} \\
&= \binom{n}{k} \cdot \Bigg[p^k \cdot (1-p)^{n-k} \cdot (1-q)^{k(n-k)} \\
&\quad + \sum_{i=1}^{k-1} \binom{k}{i} \cdot p^i \cdot (1-p)^{n-i} \cdot \left(1-(1-q)^i\right)^{k-i} \cdot (1-q)^{i(n-k)}\Bigg],
\end{aligned} \tag{5.58}$$

which corresponds to the original formula of Davis and Lo (2001). Thus, with the *Infectious Defaults Model* (IDM), we obtain the following distribution of defaults:

$$F^{\mathrm{IDM}}(x) = \sum_{k=0}^{x} P_k, \tag{5.59}$$

or, in analogy to (5.51), we obtain the loss distribution

$$F^{\mathrm{IDM}}(l) = \sum_{k=0}^{\lfloor l \cdot n / LGD \rfloor} P_k. \tag{5.60}$$

The VaR can be calculated as

$$VaR_{\alpha}^{\mathrm{IDM}}(\tilde{L}) = \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(l) = \frac{1}{n} \cdot LGD \cdot \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(x), \tag{5.61}$$

and the ES can be computed using the definition of the ES (2.19).

The main problem for an application of (5.60) is to determine the probability of a direct default p and the infection probability q. Usually, statistical models only provide the (combined) probability of default PD without separating these types of defaults. Thus, if the infection probability q could be determined exogenously, it is plausible to demand that the probability p shall be consistent with the estimation of PD with respect to the expected number of defaults:[311]

$$\mathbb{E}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right) = n \cdot \left(1 - (1-p) \cdot (1 - p \cdot q)^{n-1}\right) \stackrel{!}{=} n \cdot PD. \tag{5.62}$$

Consequently, the remaining task is to find a method to estimate q from historical or market data. Unfortunately, this problem could not be solved by Davis and Lo (2001). Thus, for the time being it seems necessary to rely on the opinion of experts which infection probabilities seem to be reasonable for a specific portfolio or sector.

5.2.4.3 Integrating Infectious Defaults into the BET-Model

Setup of the Model

As demonstrated by Düllmann (2006), the BET-model can significantly underestimate the VaR if the asset returns of the credits are positively correlated. Thus, the BET-model seems not suitable for measuring concentration risk, which is usually characterized by a high degree of default interaction. Against this background, Düllmann (2006) combines the infection model of Davis and Lo

[311] The expected number of defaults in the infectious defaults model is determined in Appendix 5.5.7.

(2001), which explicitly considers default interaction, with the BET-model. For this purpose, at first a heterogeneous portfolio is mapped into a comparable homogeneous portfolio as in the BET-model. Thus, the average probability of default $\overline{p}$ as well as the Diversity Score D are calculated according to (5.45) and (5.46). Using this hypothetical portfolio consisting of D credits, the default distribution is calculated on the basis of the infectious defaults model, leading to the following expression for the VaR in the *infection model* (IM) of Düllmann:[312]

$$VaR_{\alpha}^{\mathrm{IM}}\left(\tilde{L}\right) = \frac{1}{D} \cdot LGD \cdot \left(F_{\alpha}^{\mathrm{IDM}}\right)^{-1}(x), \tag{5.63}$$

with the distribution function $F_{\alpha}^{\mathrm{IDM}}$ of the infectious defaults model from (5.59). At this point, the probabilities of a direct default p and indirect default q are still required as additional input parameters. Similar to the suggestion of Davis and Lo (2001) to choose the parameter p for a given parameter q in a way that the expected loss is correct, Düllmann (2006) proposes to choose the parameters in a way that the VaR is identical to the "true" VaR of a multi-factor model. For this purpose, he determines the VaR at confidence level 0.999 with Monte Carlo simulations and chooses the parameter q for a given parameter p that solves the following equation:

$$VaR_{0.999}^{\mathrm{IM}}\left(\tilde{L}\right) \overset{!}{=} VaR_{0.999}^{\mathrm{MC}}\left(\tilde{L}\right). \tag{5.64}$$

In principle, it is possible not only to match the VaR but also to match the EL. In this case, both parameters p and q would be a result of these two conditions. Instead, Düllmann (2006) uses only condition (5.64) and uses the value of the averaged PD for the parameter p. Since the direct defaults should actually be only a part of the total number of defaults, the expectation of the loss distribution is too high when using this approach. However, if only the VaR is of interest, this procedure should be sufficient.[313]

The next steps are very similar to the procedure of Cespedes et al. (2006). At first, the VaR is computed for a multitude of portfolios and the corresponding infection probabilities q are determined. Then, the infection probability is explained by several portfolio variables with a linear regression. For this purpose, Düllmann (2006) chooses the following regression model:

$$\ln(q) = a_0 + a_1 \cdot \ln(HHI) + a_2 \cdot \ln(\overline{p}) + a_3 \cdot \ln(\overline{r}_{\mathrm{Intra}}) + a_4 \cdot \ln(\overline{r}_{\mathrm{Inter}}) + \varepsilon, \tag{5.65}$$

where the explanatory variables shall explain most of the dependence structure. The Herfindahl–Hirschmann index HHI is calculated as the sum of squared relative exposure shares of the sectors in the portfolio, which is similar to the definition used by Cespedes et al., who rely on the share of regulatory Pillar 1 capital instead of the

[312]Cf. (5.61) for the corresponding expression without using the parameters of the BET-model.

[313]Düllmann (2006) mentions that the simultaneous computation of both parameters leads to numerical problems. For this reason, the discrepancy in the EL is accepted. Cf. Düllmann (2006), p. 10.

share of exposure. The average probability of default $\overline{p}$ is calculated with (5.45). The variables $\overline{r}_{\text{Intra}}$ and $\overline{r}_{\text{Inter}}$ are the average intra- and inter-sector correlations, which are weighted by the total exposure amounts of the corresponding sectors. Thus, the calculation is similar to the average weighted inter-sector correlation $\overline{\beta}$ from (5.37), except for the weighting with the total exposure instead of the regulatory capital under Pillar 1. In this context, it is important to notice that Düllmann (2006) uses a definition of the sector correlations that is slightly different from the definition used in the preceding sections. While we use the term inter-sector correlation for the correlation between the sector factors, Düllmann (2006) uses this expression for the correlation between the asset returns of two borrowers, which belong to different sectors, leading to

$$\operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right) = \begin{cases} 1 & \text{if } s = t \text{ and } i = j, \\ \overline{r}_{\text{Intra}} & \text{if } s = t \text{ and } i \neq j, \\ \overline{r}_{\text{Inter}} & \text{if } s \neq t, \end{cases} \tag{5.66}$$

which already takes into account that the correlation parameters are assumed to be homogeneous. Thus, the relation between "our" intra- and inter-sector correlation ρ_{Intra} and ρ_{Inter} and "Düllmann's" correlation parameters $\overline{r}_{\text{Intra}}$ and $\overline{r}_{\text{Inter}}$ is

$$\overline{r}_{\text{Intra}} = \rho_{\text{Intra}} \quad \text{and} \quad \overline{r}_{\text{Inter}} = \rho_{\text{Intra}} \cdot \rho_{\text{Inter}} \tag{5.67}$$

in a homogeneous setting.[314] The coefficients $a_0, \ldots, a_4$ of regression model (5.65) are estimated using the ordinary least squares (OLS) technique. Finally, after application of the resulting regression function, the VaR can be approximated for any credit portfolio by computation of (5.63).

Calibration and Implementation of the Model

For the calibration of the model, several portfolios are constructed which differ in the degree of concentration, the PDs, and the correlation coefficients.[315] It is assumed that the portfolio consists of 2,000 credits with identical exposure size. In the first of four portfolio types there are only three different sectors with a sectoral exposure weight of 50%, 30%, and 20%. This leads to a HHI of 38%. The second portfolio is constructed from the first one by splitting each sector into two new sectors, where the first one has a share of 2/3 and the second one of 1/3. The same procedure is repeated for the third and the fourth portfolio type, so that the last portfolio consists of $3 \cdot 2^3 = 24$ sectors and contains the smallest sector concentration with a HHI of 6.5%. In addition to this variation, the probability of default is

[314] See also definition (5.4) of Sect. 5.2.1.

[315] The portfolios used for calibration correspond to the setting of Düllmann (2006).

Table 5.4 Parameter combinations for the calibration of the model

PD (%)	$\bar{r}_{\text{Intra}}$(%)	$\bar{r}_{\text{Inter}}$(%)		
0.03	5.0	2.5		
0.20	10.0	2.5	5.0	
0.50	15.0	2.5	5.0	7.5
1.00	20.0	5.0	7.5	10.0
2.00	30.0	5.0	10.0	15.0
5.00	40.0	5.0	10.0	15.0

varied between 0.03% and 5%, the correlation parameter $\bar{r}_{\text{Intra}}$ between 5% and 40%, and the correlation parameter $\bar{r}_{\text{Inter}}$ between 2.5% and 15%.[316] These parameters are identical for every credit of a specific portfolio. Thus, for each of the four mentioned portfolio types the parameter combinations shown in Table 5.4 are applied, leading to 360 portfolios in total.

Consistent with the preceding sections, we implement the ES instead of the VaR. Thus, for each of these portfolios, the ES is computed on the basis of a standard Monte Carlo simulation. Within the calculation of ES in the infection model, the value of the averaged PD is used for the parameter p as noticed before. Then, the infection probability q is determined, which leads to a match between the ES of the infection model and the Monte Carlo simulation:

$$ES_{0.999}^{\text{IM}}\left(\tilde{L}\right) \stackrel{!}{=} ES_{0.999}^{\text{MC}}\left(\tilde{L}\right). \tag{5.68}$$

When determining the ES in the infection model, we have to calculate the inverse CDF $(F_{\alpha}^{\text{IDM}})^{-1}$ with (5.59) and the Diversity Score D with (5.46), which requires the default correlation of (5.47). The computation of D can be quite time-consuming but the calculation can be accelerated significantly. Looking at the Diversity Score

$$D = \frac{\bar{p}\cdot(1-\bar{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \text{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}, \tag{5.69}$$

we find that the calculation requires n^2-times the calculation of the denominator, with $\sum_{s=1}^{S} n_s = n$ for the total number of credits, and especially n^2-times the calculation of the default correlation.[317] Similar to the computation of the multi-factor adjustment from Sect. 5.2.2.2, building PD-classes for each sector can reduce the calculation time notably. With N_{PD} for the number of PD-classes, we can build $S \cdot N_{PD} =: B$ different buckets with a number of n_u credits in each bucket u

[316] Due to the characteristic of the correlation parameter (5.67), the parameter $\bar{r}_{\text{Inter}}$ is always smaller than the parameter $\bar{r}_{\text{Intra}}$.

[317] See (5.47).

$(\sum_{u=1}^{B} n_u = n)$. Thus, a bucket u corresponds to all credits in a specific combination of a sector and a PD-class. Using this notation, the denominator of D can be written as

$$\begin{aligned}\frac{\overline{p}\cdot(1-\overline{p})}{D} &= \sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \mathrm{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) \\ &\quad \cdot\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)} \\ &= \sum_{u=1}^{B}\sum_{v=1}^{B} w_u\cdot w_v\cdot \mathrm{Corr}\left(1_{\{\tilde{D}_u\}}, 1_{\{\tilde{D}_v\}}\right)\cdot\sqrt{PD_u(1-PD_u)PD_v(1-PD_v)} \\ &\quad +\sum_{u=1}^{B}\sum_{i=1}^{n_B} {w_{u,i}}^2\cdot\left(1-\mathrm{Corr}\left(1_{\{\tilde{D}_u\}}, 1_{\{\tilde{D}_u\}}\right)\right)\cdot PD_u\cdot(1-PD_u).\end{aligned} \tag{5.70}$$

The first term of the resulting expression utilizes that the default correlation between creditors and the PDs are identical within each bucket. Therefore, we can sum up the corresponding terms. However, this term neglects that the asset correlation $\mathrm{Corr}(\tilde{a}_{s,i}, \tilde{a}_{s,i})$ of a credit with itself equals one. Instead, these elements are treated as if the correlation was $\mathrm{Corr}(\tilde{a}_{s,i}, \tilde{a}_{s,i}) \equiv \overline{r}_{\mathrm{Intra}}$, which is only true for $i \neq j$. Thus, we have to exchange the corresponding default correlations and set the correlation to one. This is done in the second fraction. Obviously, the computation time of (5.70) is now predominated by:[318]

$$\mathrm{Loops} = B^2 = (N_{\mathrm{PD}}\cdot S)^2. \tag{5.71}$$

Corresponding to the finding for the Pykhtin model, the number of loops does not grow with bigger portfolios. Thus, it is possible to compute the formula on bucket level within reasonable time.[319]

Using these terms, we determine the required infection probability q. As a next step, for all 360 portfolios the explanatory variables of the regression model (5.65) are calculated and the OLS-regression is performed. This leads to the following estimation function for q:

$$\begin{aligned}\ln(q) &= 0.8467 + 0.5017\cdot\ln(HHI) + 0.4726\cdot\ln(\overline{p}) \\ &\quad +1.0849\cdot\ln(\overline{r}_{\mathrm{Intra}}) + 0.6782\cdot\ln(\overline{r}_{\mathrm{Inter}}),\end{aligned} \tag{5.72}$$

[318]For the second fraction, a number of n elements has to be computed. Depending on the number of buckets or credits, the computation time can be longer than for the first term, but due to the linearity this term is virtually unproblematic.

[319]The computation time when calculating the infection model on bucket- instead on borrower-level can be reduced from 12 min to less than 1 s for a portfolio with 11 sectors, 7 PD-classes, and 5,000 creditors.

with a coefficient of determination of $R^2 = 96.7\%$. Using this formula, the infection probability and herewith the ES of every credit portfolio can be approximated very fast. If the portfolio is heterogeneous, the input parameters $\overline{p}$, $\overline{r}_{\text{Intra}}$, and $\overline{r}_{\text{Inter}}$ are the weighted averages instead of the individual parameters as described in the previous section. The performance of this model as well as the performance of the models presented in Sects. 5.2.2 and 5.2.3 will be analyzed subsequently.

5.3 Performance of Multi-Factor Models

5.3.1 Analysis for Deterministic Portfolios

To determine the quality of the presented models, we start our analysis with calculating the risk for five deterministic portfolios of different quality.[320] We generate well-diversified portfolios consisting of 5,000 credits. Consequently, we have neither high name nor high sector concentration risk. For this, we choose the sectors and their weights as given in Table 5.2. The inter-sector correlation is given in Table 5.1 whereas the intra-sector correlation is calculated on the basis of (5.8). The five portfolios differ in their PD distribution which is presented in Fig. 4.7. Portfolio 1 is the portfolio with the highest and Portfolio 5 is the one with the lowest credit quality distribution.

In Table 5.5, we compare the results from the Monte Carlo simulations (MC-Sim.), the Basel II formula (Basel II), the multi-factor adjustment of Pykhtin (Pykhtin), the formula that is based on Cespedes et al. (2006) if Monte Carlo simulations are used for calibration (CHKR I) or if the Pykhtin formula is used for the calibration (CHKR II), and the infection model of Düllmann (Düllmann). The results from the Monte Carlo simulations using the risk measure ES serve as the benchmark for the other models.

As can be seen in the table, the benchmark portfolio is constructed in a way that the Basel II formula represents a very good approximation[321] of the “real” ES in a multi-factor model given by Monte Carlo simulations.[322] Besides, the simulated VaR^{mf} matches the simulated ES^{mf}, our benchmark, almost exactly. The calculated values of the Pykhtin model are very good approximations of the ES in almost all cases, too. The outcomes of the CHKR model are somewhat more imprecise in both cases. With better credit quality, the estimation error is

[320] The results refer to the total gross loss of a portfolio in terms of ES or VaR. To relate this to the unexpected net loss, the results have to be multiplied by the LGD and the EL has to be subtracted.

[321] The small mismatch is mainly due to keeping the ES-confidence level constant and not a result of the chosen intra-sector correlation function. If we directly compare the results from Monte Carlo simulations with the ES in the ASRF framework, the relative root mean squared error is reduced from 0.97% to 0.28%.

[322] In our analyses, the number of simulation runs is 500,000.

Table 5.5 Comparison of the models for the five benchmark portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	6.23	7.68	12.95	20.88	23.15
	VaR (%)	6.18	7.62	12.94	20.93	23.30
	Absolute error (bp)	−5	−6	−1	5	15
	Relative error (%)	−0.80	−0.78	0.08	0.24	0.65
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	−11	−9	0	1	11
	Relative error (%)	−1.77	−1.17	0.00	0.05	0.48
Pykhtin	ES (%)	6.21	7.66	12.91	20.80	23.20
	Absolute error (bp)	−2	−2	−4	−8	5
	Relative error (%)	−0.32	−0.26	−0.31	−0.38	0.22
CHKR I	ES (%)	6.07	7.51	12.70	20.43	22.79
	Absolute error (bp)	−16	−17	−25	−45	−36
	Relative error (%)	−2.57	−2.21	−1.93	−2.16	−1.56
CHKR II	ES (%)	6.00	7.45	12.68	20.48	22.87
	Absolute error (bp)	−23	−23	−27	−40	−28
	Relative error (%)	−3.69	−2.99	−2.08	−1.92	−1.21
Düllmann	ES (%)	6.86	8.87	15.42	23.29	25.95
	Absolute error (bp)	63	119	247	241	280
	Relative error (%)	10.19	15.49	19.06	11.54	12.07

increasing, which leads to an underestimation of risk in high quality portfolios. However, the infection model of Düllmann shows a rather poor performance for all benchmark portfolios and overestimates the true ES significantly.

As a next step, we change the portfolio structure towards high sector concentration. For this purpose, we increase the sector weights of two sectors. We assume that 45% of the creditors – in terms of their exposure – belong to the Information Technology sector and an equal amount belongs to the Telecommunication Services sector. The remaining 10% of exposure are equally assigned to the miscellaneous sectors. As shown in Table 5.6, the risk materially increases for all types of portfolio quality. Again, the simulated values for ES^{mf} and VaR^{mf} are very close to each other. However, the Basel formula underestimates the risk by 14–20%, depending on the portfolio quality. This is the (relative) amount that should be considered in the assessment of capital adequacy under Pillar 2. The approximation formula of Pykhtin can capture this concentration risk with a negligible error in all cases. CHKR I leads to an underestimation of risk in high quality portfolios and to an overestimation of risk in low quality portfolios with a maximum deviation of nearly 4%. By contrast, in most cases the model CHKR II underestimates the risk with at maximum 6%. Thus, the sector concentration risk is not fully captured for high quality portfolios. The model of Düllmann fails to approximate the true risk and leads to a material overestimation of risk.

Furthermore, we built credit portfolios with low sector concentration. For this purpose, we use the concept of naïve diversification, implying each sector to have an equal weight of 1/11. As can be seen in Table 5.7, the economic capital is significantly lower than the regulatory capital. Moreover, this shows that it is easy

Table 5.6 Comparison of the models for five high concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	7.69	9.22	15.41	24.41	27.10
	VaR (%)	7.48	9.17	15.36	24.51	27.06
	Absolute error (bp)	–21	–5	–5	10	–6
	Relative error (%)	–2.73	–0.54	–0.32	0.41	0.15
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	–157	–163	–246	–352	–384
	Relative error (%)	–20.42	–17.68	–15.96	–14.42	–14.17
Pykhtin	ES (%)	7.66	9.29	15.46	24.39	27.03
	Absolute error (bp)	–3	7	5	–2	–7
	Relative error (%)	–0.35	0.76	0.31	–0.08	–0.24
CHKR I	ES (%)	7.40	9.08	15.59	25.07	27.95
	Absolute error (bp)	–29	–14	18	66	85
	Relative error (%)	–3.77	1.52	1.17	2.70	3.14
CHKR II	ES (%)	7.22	8.86	15.19	24.38	27.14
	Absolute error (bp)	–47	–36	–22	–3	4
	Relative error (%)	–6.11	–3.90	–1.43	–0.12	0.15
Düllmann	ES (%)	8.97	11.30	19.77	28.26	31.21
	Absolute error (bp)	128	208	436	385	411
	Relative error (%)	16.60	22.52	28.27	15.77	15.17

Table 5.7 Comparison of the models for five low concentrated portfolios with absolute error in basis points (bp) and relative error in percent (%)

		Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
MC-Sim.	ES (%)	5.66	6.98	12.16	19.78	22.06
	VaR (%)	5.64	6.94	12.17	19.81	22.10
	Absolute error (bp)	−2	–4	1	3	4
	Relative error (%)	–0.35	–0.57	0.08	0.15	0.18
Basel II	VaR (%)	6.12	7.59	12.95	20.89	23.26
	Absolute error (bp)	46	61	79	111	120
	Relative error (%)	8.13	8.74	6.50	5.61	5.44
Pykhtin	ES (%)	5.67	6.98	12.14	19.74	22.08
	Absolute error (bp)	1	0	–2	–4	2
	Relative error (%)	0.26	–0.07	–0.16	–0.21	0.09
CHKR I	ES (%)	5.66	6.94	11.92	19.17	21.38
	Absolute error (bp)	0	–4	–24	–61	–68
	Relative error (%)	0.0	–0.57	–1.97	–3.08	–3.08
CHKR II	ES (%)	5.64	6.94	12.06	19.52	21.81
	Absolute error (bp)	–2	–4	–10	–26	–25
	Relative error (%)	–0.35	–0.57	–0.82	–1.31	–1.13
Düllmann	ES (%)	5.93	7.46	13.52	21.07	23.58
	Absolute error (bp)	27	48	136	129	152
	Relative error (%)	4.71	6.95	11.19	6.51	6.90

to construct portfolios that are better diversified than the overall credit market.[323] Apart from insignificant deviations, both simulated risk measures lead to the same solutions. Again, the Pykhtin model approximates the "real" risk very good for all types of credit quality. The CHKR model I underestimates the risk for high quality portfolios with up to 3%. The CHKR model II underestimates the risk, too, but the approximation error is negligible. Again, the model of Düllmann overestimates the true risk and leads to a similar performance as the Basel II model.

5.3.2 Simulation Study for Homogeneous and Heterogeneous Portfolios

To achieve more general results, we test the models for different, randomly generated portfolios. For this reason, we implement four simulation studies. In these studies, we analyze the accuracy for homogeneous as well as for heterogeneous portfolios with respect to PD and EAD. In each simulation run, we generate a portfolio and determine its ES by the different models. After 100 runs, we calculate the root mean squared error for the outcomes of the Pykhtin model, the CHKR models I and II,[324] and the model of Düllmann in absolute and relative terms to quantify the performance of the models in comparison to Monte Carlo simulations using 500,000 trials. Furthermore, we calculate the VaR with the Basel II formula and with a Monte Carlo simulation to measure its accuracy compared to ES^{mf}. In the following, we describe the four simulation settings.

Simulation I. In this scenario, we generate portfolios with homogenous exposure sizes and homogenous PDs, that is, $w_i = 1/5000$ and $PD_i = PD = \text{const}$ for each credit. To test the accuracy for different portfolio qualities, a PD is drawn from a uniformly distribution between 0 and 10% before each new run. The sector structure and correlation is the same as in Sect. 5.2.1.

Simulation II. We generate portfolios with homogenous exposure sizes but heterogeneous PDs. For each sector, we randomly determine one of the quality distributions from Fig. 4.7. After that, we draw the PD for each credit of the sector according to this quality distribution. The exposure size remains as in Simulation I. Again, the sector structure and correlation is taken from Sect. 5.2.1.

Simulation III. We generate portfolios with homogenous PDs as in Simulation I but with heterogeneous exposure sizes. Firstly, we randomly choose the number of sectors between 2 and 11. Then, we apply a uniform distribution between 0 and 1 for the weight of every sector and scale this such that the weights sum up to one. The weights for the credits in each sector are determined in the same manner. The correlations remain unchanged.

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[323]If we consider all 25,000 simulated portfolios from Sect. 5.2.3, the lowest measured economic capital requirement was even 26% lower than the regulatory capital. This underlines the prospects of actively managing credit portfolios, e.g. with credit derivatives, but this is not in the scope of this thesis.

[324]CHKR I still corresponds to the *DF*-function based on Monte Carlo simulation and CHKR II on the Pykhtin formula.

Simulation IV. In this setting, the PDs as well as the exposure sizes of the generated portfolios are heterogeneous. The PDs are determined as in Simulation II and the exposure sizes as in Simulation III.

In each simulation, we calculate the intra-sector correlations with (5.8) and choose 5,000 credits. These portfolios contain a relatively low amount of name concentration. Instead, we focus on sector concentration. The reason is that the identical methodology for measuring name concentrations, the granularity adjustment, can be used within all implemented approaches. Thus, we prefer to avoid name concentrations to be able to separately analyze the effect of sector concentrations. The degree of sector concentration differs between the simulations. In Simulation I and II, the portfolios consist of homogenous exposures, so their HHI is in each case $1/11 = 9.1\%$. This equals the value for a naïve diversified portfolio. On the contrary, in Simulation III and IV exposures are chosen randomly and the HHI of the generated portfolios can take values between 9.1% (naïve diversification) and 1 (perfect concentration). The mean of these HHIs is around 30% in each simulation, which is only slightly higher than the HHIs of the bank portfolios analyzed by Acharya et al. (2006), which shows that the setting leads to a realistic degree of diversification.[325] The results of our simulation study can be found in Table 5.8.

Again, the outcomes of the Pykhtin model are good approximations of the "true" ES calculated with Monte Carlo simulations. Especially, if EADs are heterogeneous (simulation setting III and IV), the results are very good. Both types of the CHKR

Table 5.8 Accuracy of different models in comparison with the "true" ES calculated with Monte Carlo simulations for the specified simulation studies

		Simulation Setting I	Simulation Setting II	Simulation Setting III	Simulation Setting IV
MC-Sim. VaR	Ø Absolute error (bp)	18	6	22	8
	Ø Relative error (%)	0.67	0.43	0.77	0.60
Basel II	Ø Absolute error (bp)	259	186	264	379
	Ø Relative error (%)	11.66	13.70	8.81	25.76
Pykhtin	Ø Absolute error (bp)	14	11	54	18
	Ø Relative error (%)	0.64	0.81	3.40	1.26
CHKR I	Ø Absolute error (bp)	54	11	47	20
	Ø Relative error (%)	1.73	0.79	1.65	1.53
CHKR II	Ø Absolute error (bp)	54	12	46	21
	Ø Relative error (%)	1.72	0.84	1.56	1.59
Düllmann	Ø Absolute error (bp)	103	185	139	224
	Ø Relative error (%)	5.84	8.58	5.84	11.28

[325] Acharya et al. (2006) examined credit portfolios of 105 Italian banks during the period 1993–1999. In this study, most bank portfolios had a HHI between 20% and 30%. However, it has to be considered that the number of different industry sectors was 23 whereas we use 11 different sectors. Thus, for a comparable degree of diversification their calculated HHI have to be slightly smaller than our HHIs.

model lead to very stable results in all simulation settings. Interestingly, the CHKR model performs even better if PDs are heterogeneous, probably because the portfolios used for calculation of the functional form have heterogeneous PDs, too, and thus the resulting portfolios are more similar. It is somewhat surprising that in Simulation III the CHKR model shows a better performance than the Pykhtin model, even if the Pykhtin formula is used for determination of the diversification factor. Probably, the approximation errors of the Pykhtin model are partially smoothed by the regression from (5.40). The results of the Düllmann model are not convincing. The model can generate better outcomes than the Basel II model but performs materially worse than the other presented models. A reason could be that the portfolios which were used for the calibration of the model are too different from the portfolios of the simulation study. Against this background, it could be interesting to repeat the calibration procedure which has been applied to the CHKR model instead of the procedure presented in Sect. 5.2.4.3 because these calibration portfolios are very similar to those used in the simulation study. Of course, this calibration would be much more time-consuming than the applied calibration if we use all 25,000 randomly generated portfolios of the CHKR calibration instead of the 360 deterministic portfolios suggested by Düllmann (2006).

The comparison of the risk measures with different confidence levels shows an almost perfect match between ES^{mf} and VaR^{mf}. The relative error is smaller than 1% in each case, so our simulation study clarifies that the above-mentioned theoretical problems of the VaR are not practically relevant for a very broad range of credit portfolios. Hence, there is nothing to be said against the use of the VaR for determining the credit risk from a practical point of view even if the portfolio incorporates sector concentration risk. The Basel formula, however, shows the largest inaccuracy of all tested models for any simulation. Since in simulation setting I and II a naïve diversified portfolio is taken as a basis, the Basel formula overestimates the risk in every case due to the diversification effect.

A plot of the relative errors of the Basel formula and of VaR^{mf} in simulation setting III, sorted in ascending order, can be found in Fig. 5.3. Apart from slightly higher deviations, a plot with a similar characteristics results for simulation setting IV. It can be seen that for more than 50% of the simulated portfolios the Basel VaR is too low. That means the risk measured under Pillar 1 is underestimated compared to the "real" risk. In general, this happens when the sector concentration of the generated portfolio increases, as already demonstrated for deterministic portfolios. Thus, the simulation study accentuates the need for considering sector concentration when calculating the risk of a credit portfolio. Otherwise, the risk can be massively underestimated. This conclusion coincides with that of BCBS (2006), which points out that sector concentration can increase the capital requirement up to 40%. The maximal deviation of VaR^{mf} is around 3%. Actually, for most of the generated portfolios the error is almost zero. Thus, the deviation is negligible for practical implementation. Nevertheless, in order to verify whether there is a systematic pattern, which may help to explain the occurrence of these deviations in the multi-factor setting, we have tried to find portfolio variables such as *HHI*, average correlation, or average *PD* that can explain these deviations. Since our analyses

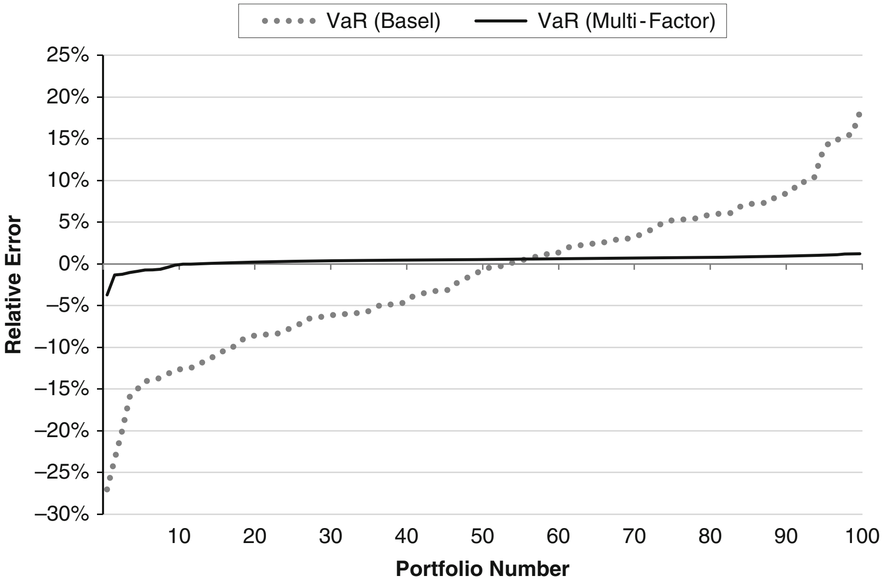

Fig. 5.3 Deviations of VaR^{Basel} and VaR^{mf} from ES^{mf}

Table 5.9 Comparison of the runtime

	Runtime: Calibration	Runtime: Application
MC-Simulation		20 min
Pykhtin		~5 s–2 min
CHKR I	30 days	0.01 s
CHKR II	150 min	0.01 s
Düllmann	240 min	~1–10 s

did not show a link between the deviations and any of the mentioned variables, it seems that the occurrence is unsystematic.

As the purpose of deriving (semi-)analytical approximation formulas for the VaR or the ES is an acceleration of the computation time, we compare the runtime of the demonstrated methods in Table 5.9.[326]

The main advantage of the Pykhtin model is that it can be applied without an excessive calibration procedure and that it is considerably faster than Monte Carlo simulations without leading to major approximation errors. The advantage of the Düllmann model is that its application is much faster but this comes at the cost of a higher approximation error. When comparing both alternative implementations of the CHKR model, we strongly propose to use the Pykhtin model for calibration (CHKR II) instead of Monte Carlo simulations (CHKR I), as the approximation

[326]The runtimes refer to a quad-core PC with 2.66 GHz CPUs (calculated on one core).

accuracy is almost identical but the computation time for determination of the *DF*-function is significantly lower. As this calibration procedure has to be computed only once for a specified correlation structure and the application of the formula is very fast, in most situations the CHKR type model should be a very good choice.

5.4 Interim Result

In this chapter, we have proposed a methodology to perform multi-factor models that are able to measure concentration risk in credit portfolios in terms of economic capital. In contrast to the existing literature regarding concentration risk, this procedure delivers results that are consistent with Basel II and has the advantage of quite low data requirements since the intra-sector correlation does not have to be estimated from historical bank data. Furthermore, we have applied this methodology to different multi-factor approaches. Since the calibration or application of these approaches is quite time-consuming for large portfolios in the original settings, which is one of the main problems of these approaches, we have demonstrated how these calculations can be accelerated significantly. As a next step, we have compared the performance of these approaches within a simulation study as the capability of different models to measure sector concentration risk has only been tested in a rather brief analysis of Düllmann (2007) before. It could be shown that it is possible to achieve good approximations in reasonable time if the approaches are adjusted in the proposed way. We have also analyzed whether the theoretical shortcomings of the Value at Risk, which can arise when leaving the ASRF framework, lead to undesirable results. Although it is indisputable that the ES has theoretical advantages over the VaR, which has already been demonstrated in several contrived portfolio examples, our framework seems well suited to explore this question for a variety of more realistic credit portfolios. We find that the accuracy of the VaR turns out to be almost perfect compared to the ES for a multitude of generated portfolios. Therefore, in our opinion, it is unproblematic to use the VaR for measuring sector concentration risk in credit portfolios.

During the specification of the multi-factor setting, we have determined input parameters, especially the inter- and intra-sector correlations, in a way that the results are comparable with the regulatory Pillar 1 capital. Thus, we do not follow some approaches that assume a pure diversification effect compared with the Basel II formula. Instead, we relate the results to a well-diversified portfolio as assumed when calibrating the Basel II formula and determine a function for the implied intra-sector correlation. Hence, it is possible to directly consider the extent of credit risk concentrations in the assessment of capital adequacy under Pillar 2. Using these modifications, we have performed an extensive numerical study similar to Cespedes et al. (2006) to get a closed form approximation formula and show how the calibration can be accelerated significantly without worsening the accuracy. In addition, we suggest computing the multi-factor adjustment and the infection model on a bucket instead of a borrower level. This allows computing the formulas of Pykhtin (2004) as well as the formulas of Düllmann (2006) much faster than Monte Carlo simulations even for a high number of credits. Moreover, due to the theoretical advantages of ES, we have determined the approximation formulas for our modified variants of Cespedes et al. (2006) and Düllmann (2006) using the risk measure ES instead of the VaR.

Having assured Basel II consistent capital requirements, we have analyzed the impact of credit concentration risk and have carried out a simulation study to compare the performance of the (modified) models from Pykhtin (2004), Cespedes et al. (2006), and Düllmann (2006). We find that the Pykhtin model leads to very good results for homogeneous as well as heterogeneous PDs if EADs are homogeneous. The performance is slightly lower for heterogeneous EADs. The results of the Cespedes-type model have a throughout high accuracy. Interestingly, the approach works better for heterogeneous portfolios. In comparison, the model of Düllmann (2006) performs rather poorly. In general, the models of Pykhtin (2004) as well as the Cespedes-type model are both well-suited for approximating the economic capital in a multi-factor setting when adjusted in the proposed way. The main advantage of the Pykhtin model is that it can directly be applied to an arbitrary portfolio type, whereas the Cespedes-type approach should not be used without initially performing the demonstrated extensive numerical work if the portfolio structure is very different. On the contrary, the results of the Cespedes-type model are slightly better for heterogeneous portfolios and it allows for ad-hoc analyses including sensitivity analyses when the non-recurring extensive numerical work is progressed.

5.5 Appendix

5.5.1 *Optimal Choice of the Single Correlation Factor*

To relate $\tilde{\bar{L}}$ to $\tilde{L}$, it is assumed that the new systematic factor $\tilde{\bar{x}}$ has a linear dependence to the original sector factors:[327]

$$\tilde{\bar{x}} = \sum_{k=1}^{K} b_k \cdot \tilde{z}_k, \tag{5.73}$$

$$\text{with} \quad \sum_{k=1}^{K} b_k^2 = 1. \tag{5.74}$$

Condition (5.74) satisfies that the new systematic factor still has a variance of 1. In order to specify the correlation factors c_i and the coefficients b_k, it will be required that the loss $\tilde{\bar{L}}$ equals the conditional expectation of the "true" loss $\mathbb{E}(\tilde{L}|\tilde{\bar{x}})$.

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[327]In contrast to this representation, Pykhtin (2004) applies these and the following formulas to n sector factors whereas we use K sector factors with $K \leqslant n$. This can lead to a significant reduction of the computation time as will be shown later on.

This assures that the first element of the subsequently performed Taylor series expansion vanishes.[328] To determine $\mathbb{E}(\tilde{\tilde{L}}|\tilde{\tilde{x}})$, we first recall that the asset return of obligor i in sector s can be written as

$$\tilde{a}_{s,i} = \sqrt{\rho_{\mathrm{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\mathrm{Intra},i}} \cdot \tilde{\xi}_i. \tag{5.75}$$

Now, each original sector factor $\tilde{x}_s$ is decomposed into a part that is related to the single-factor $\tilde{\tilde{x}}$ and a part that is independent of this factor:

$$\tilde{x}_s = \overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{1 - \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s, \tag{5.76}$$

with $\tilde{\eta}_s \sim \mathcal{N}(0,1)$. Using (5.2), (5.73), and the independence of $\tilde{z}_i, \ \tilde{z}_j$ if $i \neq j$, the correlation parameter $\overline{\rho}_s$ can be expressed as

$$\begin{aligned} \overline{\rho}_s = \mathrm{Corr}\left(\tilde{x}_s, \tilde{\tilde{x}}\right) &= \mathrm{Corr}\left(\sum_{k=1}^{K} \alpha_{s,k} \cdot \tilde{z}_k, \sum_{k=1}^{K} b_k \cdot \tilde{z}_k\right) \\ &= \sum_{k=1}^{K} \alpha_{s,k} \cdot b_k \cdot \mathbb{V}(\tilde{z}_k) = \sum_{k=1}^{K} \alpha_{s,k} \cdot b_k. \end{aligned} \tag{5.77}$$

Using this notation, the asset return (5.75) can now be written as

$$\begin{aligned} \tilde{a}_{s,i} &= \sqrt{\rho_{\mathrm{Intra},i}} \cdot \tilde{x}_s + \sqrt{1 - \rho_{\mathrm{Intra},i}} \cdot \tilde{\xi}_i \\ &= \sqrt{\rho_{\mathrm{Intra},i}} \cdot \left(\overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{1 - \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s\right) + \sqrt{1 - \rho_{\mathrm{Intra},i}} \cdot \tilde{\xi}_i \\ &= \sqrt{\rho_{\mathrm{Intra},i}} \cdot \overline{\rho}_s \cdot \tilde{\tilde{x}} + \sqrt{\rho_{\mathrm{Intra},i} - \rho_{\mathrm{Intra},i} \cdot \overline{\rho}_s^{\,2}} \cdot \tilde{\eta}_s + \sqrt{1 - \rho_{\mathrm{Intra},i}} \cdot \tilde{\xi}_i. \end{aligned} \tag{5.78}$$

The independent standard normally distributed random variables $\tilde{\eta}_s$ and $\tilde{\xi}_i$ can be combined into a new standard normally distributed random variable $\tilde{\zeta}_i$, leading to

$$\tilde{a}_{s,i} = \sqrt{\rho_{\mathrm{Intra},i}} \cdot \overline{\rho}_i \cdot \tilde{\tilde{x}} + \sqrt{1 - \left(\sqrt{\rho_{\mathrm{Intra},i}} \cdot \overline{\rho}_i\right)^2} \cdot \tilde{\zeta}_i, \tag{5.79}$$

with $\overline{\rho}_i = \overline{\rho}_s$ for each obligor i in sector s. Since the variable $\tilde{\zeta}_i$ is independent of $\tilde{\tilde{x}}$, we can use the known formula of the single-factor model for the conditional expectation

$$\mathbb{E}\left(\tilde{L}|\tilde{\tilde{x}}\right) = \sum_{i=1}^{n} w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_{\mathrm{Intra},i}} \cdot \overline{\rho}_i \cdot \Phi^{-1}(\alpha)}{\sqrt{1 - \left(\sqrt{\rho_{\mathrm{Intra},i}} \cdot \overline{\rho}_i\right)^2}}\right]. \tag{5.80}$$

[328] This simplification of the Taylor series could already be used for the granularity adjustment in Sect. 4.2.1.1.

The mentioned condition $\tilde{\tilde{L}} = \mathbb{E}(\tilde{L}|\tilde{\tilde{x}})$ leads to

$$\begin{aligned}
&\tilde{\tilde{L}} = \mathbb{E}(\tilde{L}|\tilde{\tilde{x}}) \\
&\Leftrightarrow \Phi\left[\frac{\Phi^{-1}(PD_i) - c_i \cdot \tilde{\tilde{x}}}{\sqrt{1 - c_i{}^2}}\right] = \Phi\left[\frac{\Phi^{-1}(PD_i) - \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i \cdot \Phi^{-1}(\alpha)}{\sqrt{1 - \left(\sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i\right)^2}}\right] \\
&\Leftrightarrow c_i = \sqrt{\rho_{\text{Intra},i}} \cdot \overline{\rho}_i = \sqrt{\rho_{\text{Intra},i}} \cdot \sum_{k=1}^{K} \alpha_{i,k} \cdot b_k,
\end{aligned} \tag{5.81}$$

using (5.9), (5.80), (5.77), and $\alpha_{i,k} = \alpha_{s,k}$ for each obligor i in sector s. While $\rho_{\text{Intra},i}$ and $\alpha_{i,k}$ are known, the coefficients b_k are unknown.

While (5.81) already satisfies that the first-order term of the Taylor series vanishes, the concrete choice of the parameter set $\{b_k\}$ is critical concerning the distance between the zeroth-order term $q_\alpha(\tilde{\tilde{L}})$ and the unknown quantile $q_\alpha(\tilde{L})$. Unfortunately, it is not obvious how this distance can be minimized. Thus, Pykhtin (2004) relies on the intuition that coefficients which maximize the (weighted) correlation between the single factor $\tilde{\tilde{x}}$ and the sector factors $\{\tilde{x}_s\}$ should lead to good results. This leads to the following maximization problem:

$$\max_{\{b_k\}} \left(\sum_{i=1}^{n} d_i \cdot \overline{\rho}_i\right) = \max_{\{b_k\}} \left(\sum_{i=1}^{n} d_i \cdot \sum_{k=1}^{K} \alpha_{i,k} \cdot b_k\right), \tag{5.82}$$

subject to

$$\sum_{k=1}^{K} b_k^2 = 1. \tag{5.83}$$

The solution of this optimization problem is[329]

$$b_k = \sum_{i=1}^{n} \frac{d_i \cdot \alpha_{ik}}{2\tau}, \tag{5.84}$$

where the positive constant Lagrange multiplier τ is chosen in a way that $\{b_k\}$ satisfies the constraint. As a final step, the weighting factor d_i has to be chosen. After trying several specifications, Pykhtin (2004) uses

$$d_i = w_i \cdot LGD_i \cdot \Phi\left[\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_{\text{Intra},i}} \cdot \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_{\text{Intra},i}}}\right], \tag{5.85}$$

[329] Cf. Pykhtin (2004).

which is the VaR formula in a single-factor model. The intuition behind this choice is that obligors with a high exposure in terms of VaR should have a large weight in the maximization problem whereas obligors with a small VaR should have a minor impact. Summing up, the correlation parameter c_i results from (5.81), where the coefficients b_k are determined by (5.83)–(5.85).

5.5.2 Conditional Correlation

The correlation conditional on $\tilde{\bar{x}}$ between the asset returns from (5.19) can be written as

$$\begin{aligned}
\rho_{ij}^{\bar{x}} &= \operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j} | \tilde{\bar{x}}\right) \\
&= \frac{\operatorname{Cov}\left(\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}} \cdot \alpha_{s,k} - c_i \cdot b_k\right) \cdot \tilde{z}_k, \sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},j}} \cdot \alpha_{t,k} - c_j \cdot b_k\right) \cdot \tilde{z}_k\right)}{\sqrt{\mathbb{V}\left(\tilde{a}_{s,i}|\tilde{\bar{x}}\right)} \cdot \sqrt{\mathbb{V}\left(\tilde{a}_{t,j}|\tilde{\bar{x}}\right)}} \\
&= \frac{\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}} \cdot \alpha_{s,k} - c_i \cdot b_k\right) \cdot \left(\sqrt{\rho_{\text{Intra},j}} \cdot \alpha_{t,k} - c_j \cdot b_k\right)}{\sqrt{1 - c_i{}^2} \cdot \sqrt{1 - c_j{}^2}},
\end{aligned} \tag{5.86}$$

using the independence of the factors $\tilde{z}_k$. The numerator can be simplified using $\sum_{k=1}^{K} \alpha_{s,k} \cdot b_k = c_i \big/ \sqrt{\rho_{\text{Intra},i}}$ from (5.81) and $\sum_{k=1}^{K} b_k^2 = 1$ from (5.74):

$$\begin{aligned}
&\sum_{k=1}^{K}\left(\sqrt{\rho_{\text{Intra},i}} \cdot \alpha_{s,k} - c_i \cdot b_k\right) \cdot \left(\sqrt{\rho_{\text{Intra},j}} \cdot \alpha_{t,k} - c_j \cdot b_k\right) \\
&\quad = \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - \sqrt{\rho_{\text{Intra},i}} \cdot c_j \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot b_k \\
&\qquad - \sqrt{\rho_{\text{Intra},j}} \cdot c_i \cdot \sum_{k=1}^{K} \alpha_{t,k} \cdot b_k + c_i \cdot c_j \cdot \sum_{k=1}^{K} b_k{}^2 \\
&\quad = \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - \sqrt{\rho_{\text{Intra},i}} \cdot c_j \cdot \frac{c_i}{\sqrt{\rho_{\text{Intra},i}}} \\
&\qquad - \sqrt{\rho_{\text{Intra},j}} \cdot c_i \cdot \frac{c_j}{\sqrt{\rho_{\text{Intra},j}}} + c_i \cdot c_j \\
&\quad = \sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - c_j \cdot c_i.
\end{aligned} \tag{5.87}$$

This leads to

$$\rho_{ij}^{\bar{x}} = \frac{\sqrt{\rho_{\text{Intra},i}} \cdot \sqrt{\rho_{\text{Intra},j}} \cdot \sum_{k=1}^{K} \alpha_{s,k} \cdot \alpha_{t,k} - c_i \cdot c_j}{\sqrt{1 - c_i{}^2} \cdot \sqrt{1 - c_j{}^2}}. \tag{5.88}$$

5.5.3 Calculation of the Decomposed Variance

In order to determine the conditional variance, it is decomposed into the following terms:[330]

$$\mathbb{V}\left(\tilde{L}|\tilde{\bar{x}} = \bar{x}\right) = \mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}} = \bar{x}\right] + \mathbb{E}\left[\mathbb{V}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}} = \bar{x}\right]. \tag{5.89}$$

For calculation of these terms, first the expressions (a) $\mathbb{E}(\tilde{L}|\{\tilde{z}_k\})$, (b) $\mathbb{E}(\tilde{L}^2|\{\tilde{z}_k\})$, and (c) $\mathbb{V}(\tilde{L}|\{\tilde{z}_k\})$ will be calculated. The conditional loss is given as

$$\tilde{L}|\{\tilde{z}_k\} = \sum_i w_i \cdot \left(\widetilde{LGD}_i|\{\tilde{z}_k\}\right) \cdot \left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right), \tag{5.90}$$

and for stochastically independent LGDs this leads to

$$\tilde{L}|\{\tilde{z}_k\} = \sum_i w_i \cdot \widetilde{LGD}_i \cdot \left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right). \tag{5.91}$$

(a) With $\mathbb{E}\left(\widetilde{LGD}_i\right) =: ELGD_i$ and $\mathbb{E}\left(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right) =: p_i(\{\tilde{z}_k\})$ we obtain:

$$\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right) = \sum_i w_i \cdot ELGD_i \cdot p_i(\{\tilde{z}_k\}). \tag{5.92}$$

(b) Consider that $1^2_{\{\tilde{D}_i\}} = 1_{\{\tilde{D}_i\}}$, $\mathbb{E}\left(\widetilde{LGD}^2\right) = \mathbb{E}^2\left(\widetilde{LGD}\right) + \mathbb{V}\left(\widetilde{LGD}\right) =: ELGD^2 + VLGD$, and

$$\begin{aligned} \mathbb{E}\left(LGD_i LGD_j\right) &= \text{Cov}\left(LGD_i, LGD_j\right) + \mathbb{E}(LGD_i)\mathbb{E}\left(LGD_j\right) \\ &= \mathbb{E}(LGD_i)\mathbb{E}\left(LGD_j\right) \\ &=: ELGD_i ELGD_j, \end{aligned} \tag{5.93}$$

[330]The following calculations are based on Tasche (2006a), p. 41 ff.

as well as

$$
\begin{aligned}
\mathbb{E}\Big(1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\Big) &= \mathrm{Cov}\Big(1_{\{\tilde{D}_i\}}, 1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\Big) + \mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\Big)\mathbb{E}\Big(1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\Big) \\
&= \mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\Big)\mathbb{E}\Big(1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\Big) \\
&= p_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\}).
\end{aligned}
\tag{5.94}
$$

Moreover, we have

$$
\left(\sum_i x_i\right)^2 = \sum_i\sum_j x_ix_j = \sum_i x_i{}^2 + \sum_i\sum_{j\neq i} x_ix_j, \tag{5.95}
$$

$$
\sum_{j\neq i} x_ix_j = \sum_j x_ix_j - x_i{}^2. \tag{5.96}
$$

Thus, we obtain:

$$
\begin{aligned}
\mathbb{E}\Big(\tilde{L}^2|\{\tilde{z}_k\}\Big) &= \mathbb{E}\left[\sum_i \Big(w_i\widetilde{LGD}_i 1_{\{\tilde{D}_i\}}\Big)^2|\{\tilde{z}_k\}\right] \\
&= \mathbb{E}\left[\sum_i w_i{}^2\widetilde{LGD}_i{}^2 1^2_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right] \\
&\quad + \mathbb{E}\left[\sum_i\sum_{j\neq i} w_iw_j\widetilde{LGD}_i\widetilde{LGD}_j 1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}|\{\tilde{z}_k\}\right] \\
&= \sum_i w_i{}^2\mathbb{E}\big(LGD_i{}^2\big)p_i(\{\tilde{z}_k\}) \\
&\quad + \sum_i\sum_{j\neq i} w_iw_j\mathbb{E}\big(LGD_iLGD_j\big)\mathbb{E}\Big(1_{\{\tilde{D}_i\}}1_{\{\tilde{D}_j\}}\Big) \\
&= \sum_i w_i{}^2\big(ELGD_i{}^2 + VLGD_i\big)p_i(\{\tilde{z}_k\}) \\
&\quad + \sum_i\sum_{j\neq i} w_iw_jELGD_iELGD_jp_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\}) \\
&= \sum_i w_i{}^2\big(ELGD_i{}^2 + VLGD_i\big)p_i(\{\tilde{z}_k\}) - \sum_i w_i{}^2ELGD_i{}^2p_i{}^2(\{\tilde{z}_k\}) \\
&\quad + \sum_i\sum_j w_iw_jELGD_iELGD_jp_i(\{\tilde{z}_k\})p_j(\{\tilde{z}_k\}) \\
&= \sum_i w_i{}^2\Big(ELGD_i{}^2\big[p_i(\{\tilde{z}_k\}) - p_i{}^2(\{\tilde{z}_k\})\big] + VLGD_ip_i(\{\tilde{z}_k\})\Big) \\
&\quad + \mathbb{E}^2\big(\tilde{L}|\{\tilde{z}_k\}\big).
\end{aligned}
\tag{5.97}
$$

(c) The conditional variance $\mathbb{V}(\tilde{L}|\{\tilde{z}_k\})$ is equal to

$$\begin{aligned}\mathbb{V}\big(\tilde{L}|\{\tilde{z}_k\}\big) &= \mathbb{E}\Big(\tilde{L}^2|\{\tilde{z}_k\}\Big) - \mathbb{E}^2\big(\tilde{L}|\{\tilde{z}_k\}\big) \\ &= \sum_i w_i^2\big(ELGD_i^2\left[p_i(\{\tilde{z}_k\}) - p_i^2(\{\tilde{z}_k\})\right] + VLGD_i p_i(\{\tilde{z}_k\})\big).\end{aligned} \tag{5.98}$$

(d) Using the law of iterated expectation, we have

$$p_i(\overline{x}) = \mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\tilde{\overline{x}} = \overline{x}\Big) = \mathbb{E}\Big[\mathbb{E}\Big(1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\Big)|\overline{x}\Big] = \mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}]. \tag{5.99}$$

Thus, with (5.98) the expectation of the conditional variance can be written as

$$\begin{aligned}\mathbb{E}\left[\mathbb{V}\big(\tilde{L}|\{\tilde{z}_k\}\big)|\tilde{\overline{x}}=\overline{x}\right] &= \sum_i w_i^2\big(ELGD_i^2\big(\mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}] - \mathbb{E}\left[p_i^2(\{\tilde{z}_k\})|\overline{x}\right]\big) \\ &\quad + VLGD_i\mathbb{E}[p_i(\{\tilde{z}_k\})|\overline{x}]\big) \\ &= \sum_i w_i^2\Big(ELGD_i^2\Big(p_i(\overline{x}) - \mathbb{P}\Big[\Big(1_{\{\tilde{D}_i\}}=1\Big)\wedge\Big(1_{\{\tilde{D}_i'\}}=1\Big)|\overline{x}\Big]\Big) \\ &\quad + VLGD_i p_i(\overline{x})\big).\end{aligned} \tag{5.100}$$

For independent idiosyncratic factors $\tilde{\zeta}_i, \tilde{\zeta}_i{}' \sim \mathcal{N}(0,1)$ and with

$$p_i(\overline{x}) := \Phi\left(\frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\right) \Leftrightarrow \frac{\Phi^{-1}(PD) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}} = \Phi^{-1}(p_i(\overline{x})), \tag{5.101}$$

we get

$$\begin{aligned}&\mathbb{P}\Big[\Big(1_{\{\tilde{D}_i\}}=1\Big)\wedge\Big(1_{\{\tilde{D}_i'\}}=1\Big)|\overline{x}\Big] \\ &\quad = \mathbb{P}\Big[c_i\cdot\tilde{\overline{x}} + \sqrt{1-c_i{}^2}\cdot\tilde{\zeta}_i \le \Phi^{-1}(PD_i), c_i\cdot\tilde{x} + \sqrt{1-c_i{}^2}\cdot\tilde{\zeta}_i{}' \le \Phi^{-1}(PD_i)|\overline{x}\Big] \\ &\quad = \mathbb{P}\left[\tilde{\zeta}_i \le \frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}, \tilde{\zeta}_i{}' \le \frac{\Phi^{-1}(PD_i) - c_i\cdot\overline{x}}{\sqrt{1-c_i{}^2}}\right] \\ &\quad = \Phi_2\big(\Phi^{-1}(p_i(\overline{x})), \Phi^{-1}(p_i(\overline{x})), \rho_{ii}^{\overline{x}}\big),\end{aligned} \tag{5.102}$$

with the correlation conditional on $\bar{x}$ of (5.20). Hence, (5.100) results in

$$\mathbb{E}\left[\mathbb{V}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}}=\bar{x}\right]=\sum_i w_i{}^2\left(ELGD_i{}^2\left[p_i(\bar{x})-\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}(p_i(\bar{x})),\rho_{ii}^{\bar{x}}\right)\right]\right.$$
$$\left.+VLGD_i p_i(\bar{x})\right). \tag{5.103}$$

(e) Using (5.92), the variance of the conditional expectation can be expressed as

$$\begin{aligned}
&\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}}=x\right]\\
&=\mathbb{E}\left[\mathbb{E}^2\left(\tilde{L}|\{\tilde{z}_k\}\right)|\bar{x}\right]-\mathbb{E}^2\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\bar{x}\right]\\
&=\mathbb{E}\left(\mathbb{E}^2\left[\sum_i w_i\widetilde{LGD}_i 1_{\{\tilde{D}_i\}}|\{\tilde{z}_k\}\right]|\bar{x}\right)-\mathbb{E}^2\left(\sum_i w_i ELGD_i p_i(\{\tilde{z}_k\})|\bar{x}\right)\\
&=\mathbb{E}\left[\left(\sum_i w_i ELGD_i p_i(\{\tilde{z}_k\})\right)^2|\bar{x}\right]-\left(\sum_i w_i ELGD_i p_i(\bar{x})\right)^2,
\end{aligned} \tag{5.104}$$

leading to

$$\begin{aligned}
&\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}}=x\right]\\
&=\mathbb{E}\left[\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\{\tilde{z}_k\})\cdot p_j(\{\tilde{z}_k\})|\bar{x}\right]\\
&\quad-\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\bar{x})p_j(\bar{x})\\
&=\sum_i\sum_j w_i w_j ELGD_i ELGD_j \mathbb{E}\left(p_i(\{\tilde{z}_k\})\cdot p_j(\{\tilde{z}_k\})|\bar{x}\right)\\
&\quad-\sum_i\sum_j w_i w_j ELGD_i ELGD_j p_i(\bar{x})p_j(\bar{x})\\
&=\sum_i\sum_j w_i w_j ELGD_i ELGD_j\left[\mathbb{P}\left[\left(1_{\{\tilde{D}_i\}}=1\right)\wedge\left(1_{\{\tilde{D}_j\}}=1\right)|\bar{x}\right]-p_i(\bar{x})p_j(\bar{x})\right].
\end{aligned} \tag{5.105}$$

Analogous to (5.102) and using the conditional correlation (5.20), this can be expressed as:

$$\begin{aligned}
\mathbb{V}\left[\mathbb{E}\left(\tilde{L}|\{\tilde{z}_k\}\right)|\tilde{\bar{x}}=\bar{x}\right]&=\sum_i\sum_j w_i w_j ELGD_i ELGD_j\\
&\quad\cdot\left[\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}\left(p_j(\bar{x})\right),\rho_{ij}^{\bar{x}}\right)-p_i(\bar{x})p_j(\bar{x})\right].
\end{aligned} \tag{5.106}$$

5.5.4 Derivatives of the Decomposed Variance Terms

As both conditional variance terms are linear in the bivariate normal distribution, the derivative of the bivariate normal distribution will be calculated subsequently. Then, the derivatives of $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ will be computed.

Proposition. *The derivative of the bivariate normal distribution can be written as:*

$$\frac{d}{dx}\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}\left(p_j(\bar{x})\right),\rho_{ij}^{\bar{x}}\right)=\frac{dp_i(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}\left(p_j(\bar{x})\right)-\rho_{ij}^{\bar{x}}\cdot\Phi^{-1}(p_i(\bar{x}))}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)$$
$$+\frac{dp_j(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}(p_i(\bar{x}))-\rho_{ij}^{\bar{x}}\cdot\Phi^{-1}\left(p_j(\bar{x})\right)}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right). \tag{5.107}$$

Proof. Using the notation

$$y_i(\bar{x})=\frac{\Phi^{-1}(PD_i)-c_i\cdot\bar{x}}{\sqrt{1-c_i{}^2}},\ \ y_j(\bar{x})=\frac{\Phi^{-1}(PD_j)-c_j\cdot\bar{x}}{\sqrt{1-c_j{}^2}}, \tag{5.108}$$

and the chain rule, we get

$$\begin{aligned}&\frac{d}{d\bar{x}}\Phi_2\left(\Phi^{-1}(p_i(\bar{x})),\Phi^{-1}\left(p_j(\bar{x})\right),\rho_{ij}^{\bar{x}}\right)\\&=\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}),y_j(\bar{x}),\rho_{ij}^{\bar{x}}\right)\\&=\underbrace{\frac{dy_i}{d\bar{x}}}_{(I)}\underbrace{\frac{\partial}{\partial y_i}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)}_{(II)}+\underbrace{\frac{dy_j}{d\bar{x}}}_{(III)}\underbrace{\frac{\partial}{\partial y_j}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)}_{(IV)}.\end{aligned} \tag{5.109}$$

For calculation of term (II) and (IV), we rewrite the bivariate normal distribution according to Appendix 2.8.6 as

$$\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)=\int\limits_{z=-\infty}^{y_j}\varphi(z)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz. \tag{5.110}$$

Thus, we have

$$
\begin{aligned}
\frac{\partial}{\partial y_i}\Phi_2\left(y_i, y_j, \rho_{ij}^{\bar{x}}\right) &= \frac{\partial}{\partial y_i}\int\limits_{z=-\infty}^{y_j} \varphi(z)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz \\
&= \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\int\limits_{z=-\infty}^{y_j} \varphi(z)\varphi\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz \\
&= \frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\int\limits_{z=-\infty}^{y_j} \frac{1}{2\pi}\exp\left(-\frac{1}{2}\underbrace{\left[z^2+\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2\right]}_{(*)}\right)dz.
\end{aligned}
\tag{5.111}
$$

The term (*) is equivalent to

$$
\begin{aligned}
z^2+\left(\frac{y_i-\rho_{ij}^{\bar{x}}\cdot z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2 &= \frac{\left(1-\left(\rho_{ij}^{\bar{x}}\right)^2\right)z^2+{y_i}^2-2y_i\rho_{ij}^{\bar{x}}z+\left(\rho_{ij}^{\bar{x}}\right)z^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \frac{z^2-2y_i\rho_{ij}^{\bar{x}}z+{y_i}^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \frac{z^2-2y_i\rho_{ij}^{\bar{x}}z+{y_i}^2+{y_i}^2\left(\rho_{ij}^{\bar{x}}\right)^2-{y_i}^2\left(\rho_{ij}^{\bar{x}}\right)^2}{1-\left(\rho_{ij}^{\bar{x}}\right)^2} \\
&= \left(\frac{z-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2+{y_i}^2.
\end{aligned}
\tag{5.112}
$$

Hence, (5.111) can be written as

$$\frac{\partial}{\partial y_i}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)=\frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\int\limits_{z=-\infty}^{y_i}\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left[{y_i}^2+\left(\frac{z-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)^2\right]\right)dz$$
$$=\varphi(y_i)\int\limits_{z=-\infty}^{y_j}\frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\varphi\left(\frac{z-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz. \tag{5.113}$$

For solving the integral, we substitute $t := \frac{z-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-(\rho_{ij}^{\bar{x}})^2}}$, and thus $\frac{dz}{dt}=\sqrt{1-(\rho_{ij}^{\bar{x}})^2}$. This leads to

$$\frac{\partial}{\partial y_i}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)=\varphi(y_i)\int\limits_{t=-\infty}^{\frac{y_j-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-(\rho_{ij}^{\bar{x}})^2}}}\frac{1}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\varphi(t)\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}\,dt$$
$$=\varphi(y_i)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right). \tag{5.114}$$

Analogously, the term (IV) of (5.109) is equivalent to

$$\frac{\partial}{\partial y_j}\Phi_2\left(y_i,y_j,\rho_{ij}^{\bar{x}}\right)=\frac{\partial}{\partial y_j}\int\limits_{z=-\infty}^{y_i}(z)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}}z}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right)dz$$
$$=\varphi\left(y_j\right)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}y_j}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right). \tag{5.115}$$

The derivatives (I) and (III) of (5.109) are given as

$$\frac{dy_i(\bar{x})}{d\bar{x}}=-\frac{c_i}{\sqrt{1-{c_i}^2}}\quad\text{and}\quad\frac{dy_j(\bar{x})}{d\bar{x}}=-\frac{c_j}{\sqrt{1-{c_j}^2}}. \tag{5.116}$$

Thus, inserting (5.114), (5.115), and (5.116) into (5.109), the derivative of the bivariate normal distribution finally results in

$$\begin{aligned} \frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) &= -\frac{c_i}{\sqrt{1-{c_i}^2}}\varphi(y_i)\Phi\left(\frac{y_j-\rho_{ij}^{\bar{x}}y_i}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) \\ &\quad -\frac{c_j}{\sqrt{1-{c_j}^2}}\varphi\left(y_j\right)\Phi\left(\frac{y_i-\rho_{ij}^{\bar{x}}y_j}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) \\ &= \frac{dp_i(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}\left[p_j(\bar{x})\right]-\rho_{ij}^{\bar{x}}\Phi^{-1}[p_i(\bar{x})]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right) \\ &\quad +\frac{dp_j(\bar{x})}{d\bar{x}}\Phi\left(\frac{\Phi^{-1}[p_i(\bar{x})]-\rho_{ij}^{\bar{x}}\Phi^{-1}\left[p_j(\bar{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\bar{x}}\right)^2}}\right), \end{aligned} \tag{5.117}$$

where the derivatives $\frac{dp_i(\bar{x})}{d\bar{x}}$ and $\frac{dp_j(\bar{x})}{d\bar{x}}$ are given by (5.16), which is equal to proposition (5.107).

As a next step, the derivatives of $\eta_{2,c}^{\infty}(\bar{x})$ and $\eta_{2,c}^{\mathrm{GA}}(\bar{x})$ will be calculated. With

$$\begin{aligned} \eta_{2,c}^{\infty}(\bar{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} & w_i w_j ELGD_i ELGD_j \\ & \cdot\left[\Phi_2\left(\Phi^{-1}(p_i(\bar{x})), \Phi^{-1}\left(p_j(\bar{x})\right), \rho_{ij}^{\bar{x}}\right) - p_i(\bar{x})p_j(\bar{x})\right], \end{aligned} \tag{5.118}$$

we get

$$\begin{aligned} \frac{d\eta_{2,c}^{\infty}(\bar{x})}{d\bar{x}} &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \left[\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) - \frac{d}{d\bar{x}}\left(p_i(\bar{x})p_j(\bar{x})\right)\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j ELGD_i ELGD_j \\ &\quad \cdot\left[\frac{d}{d\bar{x}}\Phi_2\left(y_i(\bar{x}), y_j(\bar{x}), \rho_{ij}^{\bar{x}}\right) - \left(\frac{dp_i(\bar{x})}{d\bar{x}}p_j(\bar{x}) + \frac{dp_j(\bar{x})}{d\bar{x}}p_i(\bar{x})\right)\right]. \end{aligned} \tag{5.119}$$

Using the derivative of the bivariate normal distribution from (5.117) yields

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n}\sum_{j=1}^{n} & w_i w_j ELGD_i ELGD_j \\ & \cdot\left(\frac{dp_i(\overline{x})}{d\overline{x}}\Phi\left(\frac{\Phi^{-1}\left[p_j(\overline{x})\right]-\rho_{ij}^{\overline{x}}\Phi^{-1}[p_i(\overline{x})]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}}\right)\right. \\ & +\frac{dp_j(\overline{x})}{d\overline{x}}\Phi\left(\frac{\Phi^{-1}[p_i(\overline{x})]-\rho_{ij}^{\overline{x}}\Phi^{-1}\left[p_j(\overline{x})\right]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}}\right) \\ & \left.-\frac{dp_i(\overline{x})}{d\overline{x}}p_j(\overline{x})-\frac{dp_j(\overline{x})}{d\overline{x}}p_i(\overline{x})\right).\end{aligned} \tag{5.120}$$

Comparing the terms on the right-hand side, it can be found that the first and second summand as well as the third and fourth summand only differ concerning the indices i and j. Due to the double sum, each combination of i and j occurs twice. Thus, (5.120) can be simplified to:[331]

$$\begin{aligned}\frac{d\eta_{2,c}^{\infty}(\overline{x})}{d\overline{x}} = 2\cdot\sum_{i=1}^{n}\sum_{j=1}^{n} & w_i w_j ELGD_i ELGD_j \frac{dp_i(\overline{x})}{d\overline{x}} \\ & \cdot\left(\Phi\left(\frac{\Phi^{-1}\left[p_j(\overline{x})\right]-\rho_{ij}^{\overline{x}}\Phi^{-1}[p_i(\overline{x})]}{\sqrt{1-\left(\rho_{ij}^{\overline{x}}\right)^2}}\right)-p_j(\overline{x})\right).\end{aligned} \tag{5.121}$$

Similarly, the derivative of

$$\begin{aligned}\eta_{2,c}^{\mathrm{GA}}(\overline{x}) = \sum_{i=1}^{n} w_i{}^2 & \left(ELGD_i{}^2\left[p_i(\overline{x})-\Phi_2\left(\Phi^{-1}(p_i(\overline{x})),\Phi^{-1}(p_i(\overline{x})),\rho_{ii}^{\overline{x}}\right)\right]\right. \\ & \left.+VLGD_i p_i(\overline{x})\right)\end{aligned} \tag{5.122}$$

[331] It has to be noticed that the conditional correlation matrix is symmetric, so we have $\rho_{ij}^{\overline{x}}=\rho_{ji}^{\overline{x}}$ for all i,j.

is given as

$$\frac{d\eta_{2,c}^{\mathrm{GA}}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n} w_i^2 \left(ELGD_i^2 \left[\frac{dp_i(\overline{x})}{d\overline{x}} - \frac{d}{d\overline{x}} \Phi_2\left(y_i(\overline{x}), y_i(\overline{x}), \rho_{ii}^{\overline{x}}\right) \right] + VLGD_i \frac{dp_i(\overline{x})}{d\overline{x}} \right). \tag{5.123}$$

Inserting the derivative of the bivariate normal distribution (5.117) finally leads to

$$\frac{d\eta_{2,c}^{\mathrm{GA}}(\overline{x})}{d\overline{x}} = \sum_{i=1}^{n} w_i^2 \frac{dp_i(\overline{x})}{d\overline{x}} \cdot \left(ELGD_i^2 \left[1 - 2\Phi\left(\frac{\Phi^{-1}[p_i(\overline{x})] - \rho_{ii}^{\overline{x}} \Phi^{-1}[p_i(\overline{x})]}{\sqrt{1 - \left(\rho_{ii}^{\overline{x}}\right)^2}} \right) \right] + VLGD_i \right). \tag{5.124}$$

5.5.5 *Moment Matching in the BET-Model*

5.5.5.1 Matching the First Moment

The expected loss of the original portfolio can be calculated as

$$\mathbb{E}\left(\tilde{L}^{\mathrm{orig}}\right) = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot \mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\right) = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot PD_{s,i}, \tag{5.125}$$

and the expected loss of the hypothetical portfolio as

$$\begin{aligned} \mathbb{E}\left(\tilde{L}^{\mathrm{hyp}}\right) &= \sum_{i=1}^{D} \frac{1}{D} \cdot LGD \cdot \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \frac{1}{D} \cdot LGD \cdot \sum_{i=1}^{D} \overline{p} \\ &= \frac{1}{D} \cdot LGD \cdot D \cdot \overline{p} = LGD \cdot \overline{p}, \end{aligned} \tag{5.126}$$

with $\mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) = \overline{p}$ for all i. Thus, matching the expectation for both portfolios leads to

$$\begin{aligned} &\mathbb{E}\left(\tilde{L}^{\mathrm{orig}}\right) \overset{!}{=} \mathbb{E}\left(\tilde{L}^{\mathrm{hyp}}\right) \\ &\Leftrightarrow \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot LGD \cdot PD_{s,i} = LGD \cdot \overline{p} \\ &\Leftrightarrow \overline{p} = \sum_{s=1}^{S} \sum_{i=1}^{n_s} w_{s,i} \cdot PD_{s,i}. \end{aligned} \tag{5.127}$$

5.5.5.2 Matching the Second Moment

For the original portfolio, the variance can be calculated as

$$
\begin{aligned}
\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) &= \mathbb{V}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot LGD\cdot 1_{\{\tilde{D}_{s,i}\}}\right)\\
&= LGD^2\cdot \mathbb{V}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot 1_{\{\tilde{D}_{s,i}\}}\right)\\
&= LGD^2\cdot \operatorname{Cov}\left(\sum_{s=1}^{S}\sum_{i=1}^{n_s} w_{s,i}\cdot 1_{\{\tilde{D}_{s,i}\}}, \sum_{t=1}^{S}\sum_{j=1}^{n_t} w_{t,j}\cdot 1_{\{\tilde{D}_{t,j}\}}\right)\\
&= LGD^2\cdot \sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \operatorname{Cov}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\\
&= LGD^2\cdot \sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\\
&\quad\cdot \sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right)}\cdot \sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right)}.
\end{aligned}
\tag{5.128}
$$

As the default variable is Bernoulli distributed, the variance terms equal

$$
\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right) = PD_{s,i}\cdot\left(1-PD_{s,i}\right) \quad\text{and}\quad \mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right) = PD_{t,j}\cdot\left(1-PD_{t,j}\right)
\tag{5.129}
$$

and we obtain

$$
\begin{aligned}
\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) &= LGD^2\cdot \sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)\\
&\quad\cdot \sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}.
\end{aligned}
\tag{5.130}
$$

Due to the independence of the default events in the hypothetical portfolio, the variance of this portfolio is

$$
\begin{aligned}
\mathbb{V}\left(\tilde{L}^{\text{hyp}}\right) &= \mathbb{V}\left(\sum_{i=1}^{D}\frac{1}{D}\cdot LGD\cdot 1_{\{\tilde{D}_i\}}\right) = \frac{1}{D^2}\cdot LGD^2\cdot \mathbb{V}\left(\sum_{i=1}^{D} 1_{\{\tilde{D}_i\}}\right)\\
&= \frac{1}{D^2}\cdot LGD^2\cdot D\cdot \mathbb{V}\left(1_{\{\tilde{D}_i\}}\right) = \frac{1}{D}\cdot LGD^2\cdot \overline{p}\cdot(1-\overline{p}).
\end{aligned}
\tag{5.131}
$$

Matching the variance terms (5.130) and (5.131) leads to

$$\mathbb{V}\left(\tilde{L}^{\text{orig}}\right) \overset{!}{=} \mathbb{V}\left(\tilde{L}^{\text{hyp}}\right)$$

$$\Leftrightarrow D = \frac{\overline{p}\cdot(1-\overline{p})}{\sum_{s=1}^{S}\sum_{t=1}^{S}\sum_{i=1}^{n_s}\sum_{j=1}^{n_t} w_{s,i}\cdot w_{t,j}\cdot \operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}},1_{\{\tilde{D}_{t,j}\}}\right)\cdot\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.132}$$

5.5.6 Interrelation of the Pairwise Default Correlation and the Asset Correlation

Using the standard calculus for the correlation and covariance as well as the variance of a Bernoulli distributed variable, the pairwise default correlation between borrower i in sector s and borrower j in sector t can be expressed as

$$\begin{aligned}
\operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) &= \frac{\operatorname{Cov}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{s,i}\}}\right)}\cdot\sqrt{\mathbb{V}\left(1_{\{\tilde{D}_{t,j}\}}\right)}} \\
&= \frac{\operatorname{Cov}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{PD_{s,i}\cdot\left(1-PD_{s,i}\right)\cdot PD_{t,j}\cdot\left(1-PD_{t,j}\right)}} \\
&= \frac{\mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\cdot 1_{\{\tilde{D}_{t,j}\}}\right) - \mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\right)\cdot\mathbb{E}\left(1_{\{\tilde{D}_{t,j}\}}\right)}{\sqrt{PD_{s,i}\cdot\left(1-PD_{s,i}\right)\cdot PD_{t,j}\cdot\left(1-PD_{t,j}\right)}}.
\end{aligned} \tag{5.133}$$

The expectation values of the individual default events equal $PD_{s,i}$ and $PD_{t,j}$. Similar to (5.102), assuming a normally distributed asset return, the expectation value of a simultaneous default can be written as

$$\begin{aligned}
\mathbb{E}\left(1_{\{\tilde{D}_{s,i}\}}\cdot 1_{\{\tilde{D}_{t,j}\}}\right) &= \mathbb{P}\left[\left(1_{\{\tilde{D}_{s,i}\}} = 1\right)\wedge\left(1_{\{\tilde{D}_{t,j}\}} = 1\right)\right] \\
&= \mathbb{P}\left(\tilde{a}_{s,i} \leq \Phi^{-1}(PD_{s,i}), \tilde{a}_{t,j} \leq \Phi^{-1}(PD_{t,j})\right) \\
&= \Phi_2\left(\Phi^{-1}\left(PD_{s,i}\right), \Phi^{-1}\left(PD_{t,j}\right), \operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right)\right).
\end{aligned} \tag{5.134}$$

Thus, we get

$$\operatorname{Corr}\left(1_{\{\tilde{D}_{s,i}\}}, 1_{\{\tilde{D}_{t,j}\}}\right) = \frac{\Phi_2\left(\Phi^{-1}\left(PD_{s,i}\right), \Phi^{-1}\left(PD_{t,j}\right), \operatorname{Corr}\left(\tilde{a}_{s,i}, \tilde{a}_{t,j}\right)\right) - PD_{s,i}\cdot PD_{t,j}}{\sqrt{PD_{s,i}\left(1-PD_{s,i}\right)PD_{t,j}\left(1-PD_{t,j}\right)}}. \tag{5.135}$$

5.5.7 Expected Number of Defaults in the Infectious Defaults Model

Due to the homogeneity of the portfolio and the stochastic independence of all indicator variables, the expected number of defaults is

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} 1_{\{\tilde{D}_i\}}\right) &= n \cdot \mathbb{E}\left(1_{\{\tilde{D}_i\}}\right) \\
&= n \cdot \mathbb{E}\left(\tilde{Z}_i\right) \\
&= n \cdot \mathbb{E}\left(\tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left[1 - \prod_{j \neq i} \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)\right]\right) \\
&= n \cdot \mathbb{E}\left(\tilde{X}_i + \left(1 - \tilde{X}_i\right) \cdot \left[1 - \left(1 - \tilde{X}_j \cdot \tilde{Y}_{j,i}\right)^{n-1}\right]\right) \\
&= n \cdot \left(\mathbb{E}\left(\tilde{X}_i\right) + \left(1 - \mathbb{E}\left(\tilde{X}_i\right)\right) \cdot \left[1 - \left(1 - \mathbb{E}\left(\tilde{X}_j\right) \cdot \mathbb{E}\left(\tilde{Y}_{j,i}\right)\right)^{n-1}\right]\right) \\
&= n \cdot \left(p + (1-p) \cdot \left[1 - (1 - p \cdot q)^{n-1}\right]\right) \\
&= n \cdot \left(1 - (1-p) \cdot (1 - p \cdot q)^{n-1}\right).
\end{aligned} \tag{5.136}$$

Chapter 6
Conclusion

In the beginning of this work, it has been asserted that, despite the material relevance of concentration risk concerning the survival of banks and the stability of the whole banking system, the variety of literature and the public attention on this topic have been rather scarce. Against this background, within this work economical as well as regulatory aspects of concentration risk have been presented and some models for measuring concentration risk in credit portfolios have been explained, modified, and compared in detail. Moreover, several research questions regarding name and sector concentration risk, which have been discussed during this work, have been raised in the introduction.

In Chap. 2, the risk measures VaR and ES have been introduced, which are the most common characteristic numbers for measuring risk in credit portfolios. In this context, the emphasis has been put on the (non-)coherency and estimation issues. Then, the asset value model of Merton (1974), the one-factor model of Vasicek (1987), and the ASRF model of Gordy (2003) have been presented. These models build the fundament of the IRB Approach of Basel II, which has been explained subsequently.

In the literature and in various discussions it could be found that there are very different interpretations and characteristics of concentration risk. First of all, banks often only look at one side of concentration risk – the diversification effect. Thus, it is often argued that the requirements of Pillar 1 are the non-diversified benchmark and therefore an upper barrier for the true capital requirement. But as the Basel II formulas have been calibrated on well-diversified portfolios with low name and low sector concentrations, it is indeed possible that banks should have an additional capital buffer to capture concentration risk. Furthermore, some theoretical models as well as empirical studies have demonstrated that concentrated banks can be less risky than diversified banks, which is mainly due to better monitoring abilities of specialized financial institutions. However, even if it can be economically reasonable to be focused on particular industry sectors or geographical regions, the capital requirements should still be higher than for diversified banks. The main argument is that although a specialized bank could benefit from the ability to invest in firms with higher quality (of course it is not even clear that a higher risk-return premium is

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earned through lower risk), the bank would still be very vulnerable if the specific sector is in an economic downturn scenario. But exactly such a downturn scenario, often quantified with the VaR, plays the decisive role for the capital requirements. This point as well as regulatory requirements and industry best practices concerning the management of concentration risk have been highlighted in Chap. 3.

In Chap. 4, we have focused on the measurement of name concentrations. After presenting the first-order granularity adjustment, a second-order granularity adjustment has been derived, which results from a Taylor series expansion taking elements of higher order into account. Although during this work and in the literature it was expected that the resulting formula could improve accuracy, it has to be stated that the standard first-order granularity adjustment leads to more convincing results. As it is not analyzed sufficiently in the literature in which cases the ASRF formula leads to a convincing approximation of the true risk, we have analyzed this issue with a detailed numerical study. For this purpose, it has been determined how many credits a portfolio should at least contain if a bank intends to ignore name concentrations; this would be the case if only the ASRF formula was applied. It has been shown that the result is highly dependent on the probability of default and the asset correlation. For a high-quality portfolio, the minimum number of credits varies between 1,371 and 23,989 (A-rated), whereas the critical number of credits for a low-quality portfolio is in the bandwidth 23–205 (CCC-rated). These numbers correspond to an accepted error of 5%. The difference between high- and low-quality portfolios can be explained with a higher anticipation of unsystematic defaults for low-quality portfolios. Furthermore, we have raised the question whether the granularity adjustment is able to overcome the shortcomings of the ASRF model, which has only been analyzed rudimentarily before. The results of our study demonstrate that the granularity adjustment provides a very good approximation of the risk stemming from name concentrations. We find that a consideration of the granularity adjustment can reduce the required minimum portfolio size by on average 83.04% compared to the ASRF model.

Because of the theoretical shortcomings of the VaR and since, differently from the ASRF framework, these can be problematic if there is concentration risk, the ES has been considered, too. At a first glance, it is problematic that the ES is by definition higher than the VaR, which leads to higher capital requirements. As the change of the risk measure should solve the problem of superadditivity but should not inevitably lead to higher capital requirements, we have adjusted the confidence level of the ES in a way that the Pillar 1 formulas still lead to an almost identical level of measured risk. We find that a confidence level of $\alpha = 99.72\%$ for the ES leads to a very good concurrence between the ES and the 99.9%-VaR for all relevant credit qualities and correlations. By application of the same analyses as before for the VaR-based granularity adjustment, we find that this approach works very well. The ES-based granularity adjustment does not only reduce the required number of credits by 91.64% compared to the ASRF solution, but the minimum number of credits is also 49.05% lower compared to the VaR-based granularity adjustment. These results show that for portfolios with a significant amount of name concentrations, the ES-based granularity adjustment is really well-suited.

An additional robustness check using stochastic LGDs has confirmed these findings. However, the postulated accuracy should also be obtained in many real-world portfolios if the VaR-based granularity adjustment is applied.

In Chap. 5, we have analyzed risks stemming from sector concentrations. For this purpose, the design of multi-factor models has been explained. Since additional input parameters are needed when applying a multi- instead of a single-factor model, a methodology has been developed to parameterize intra- and inter-sector correlations consistent with the one-factor model of Pillar 1. Given the inter-sector correlation structure of the MSCI EMU industry indices, a formula for the implied intra-sector correlation has been determined. With these parameters, the results of the multi-factor model and of the Basel II formula are almost identical if the portfolio is well-diversified as it had originally been assumed when calibrating the Basel II formula. However, if the degree of concentration is higher, the capital requirement can increase significantly. Using these parameters, an extensive numerical study has been performed, which is similar to Cespedes et al. (2006). The result of our numerical study is a closed form approximation formula in a multi-factor setting, which is consistent with the Basel framework. In contrast to the resulting formula of Cespedes et al. (2006), our formula is able to measure not only the benefit from sectoral diversification but also the additional risk from sectoral concentrations if these are higher than assumed in Basel II for a typical well-diversified portfolio of large internationally active banks. Moreover, we have used the theoretically more convenient ES instead of the VaR. In addition, we have demonstrated how the extensive numerical calibration of the model can be accelerated significantly without leading to worse approximations. Using the risk measure ES, we have also performed the calibration procedure of Düllmann (2006). Furthermore, we have demonstrated how these models can be applied on bucket instead of borrower level, which accelerates the computation of the corresponding formulas considerably.

Based on the preceding findings, we have implemented our multi-factor setting and compared different models by means of a simulation study. We find that the accuracy of the models of Pykhtin (2004) and the developed formula, which is based on Cespedes et al. (2006), lead to quite good results, whereas the model of Düllmann (2006) performs rather poorly. Especially in the case of heterogeneous exposures, the model in the style of Cespedes et al. (2006) shows the best accuracy. Since the extensive numerical calibration of this model only has to be done once for a given correlation structure, and then, it is possible to perform ad-hoc analyses, this model seems to be well-suited for many real-world applications if sector concentrations shall be considered. A last very interesting result could be obtained when the VaR and the ES have been compared within the simulation study.[332] In almost all simulation runs, the relative error of the VaR compared with the ES was lower than 1%. Thus, in contrast to some contrived portfolio examples, the usage of VaR seems to be unproblematic within this more realistic setting from a practical

[332]The confidence level of the ES has been reduced to 99.72% as argued in Chap. 4.

point of view, even if there is a high degree of sector concentration risk in the portfolio.

In this work, several aspects of concentration risk have been highlighted. However, there is a variety of open issues in the context of concentration risk that could not be addressed in this work. One important topic is the consideration of concentration risk in the pricing of individual credits and credit derivatives, especially of credit portfolio derivatives like CDOs. In particular, the sensitivity of the price depending on existing risk concentrations has hardly been analyzed. Beyond that, it would be interesting to take into consideration whether a bank is exposed to the risk of a security until maturity or whether instruments of active portfolio management are employed to reduce risk concentrations. Secondly, during most of the work, it has been assumed that LGDs are deterministic or at least stochastically independent. An open issue is how portfolio risk is affected by risk concentrations stemming from collateral. In this context, concentrations in individual positions and in sectors could both have relevant effects. For example, in the financing of objects like ships or airplanes, there are usually several financiers investing in one object; hence, the impact of the individual risk component of the collateral can be even higher than that of the obligors. Similarly, in retail financing there usually is a low degree of concentration risk of the obligors but if most of a bank's loans are secured by mortgages or by cars, there can be a relevant impact of sector concentrations in collateral. Thirdly, credit contagion through micro-structural channels could only be touched upon. One challenging aspect in this area is the estimation of business relations, since micro-structural dependencies cannot be restricted to the most important firms of a bank's actual portfolio but firms that are not financed by the bank can affect the credit portfolio through their business relationships as well. Thus, additional research should address how these effects of micro-structural dependencies can be implemented in practice despite the substantial data requirements.

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